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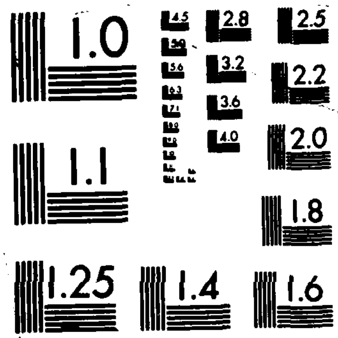

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BIFURCATIONS INTO PATHOLOGY FOR HAMILTONIAN SYSTEMS

by

Konstantin Mischaikow

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Bifurcations into Pathology for Hamiltonian Systems

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January 1986

This paper is dedicated to the memory of Charles C. Conley

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Abstract

This paper presents a geometric analysis of bifurcations leading to chaos for Hamiltonian systems with two degrees of freedom of the form

y-dot Gradient

x-dot

$\dot{x} = y$, $\dot{y} = -\nabla V(x)$. Two bifurcation parameters are considered. One is the energy level and the other is an angle, Φ, between two homoclinic orbits. Though global non-linearities are necessary, the results are obtained by local analysis of the flow near the origin where it is

D-sq

assumed that $D^2V(0) = I$, the 2×2 identity matrix. For a fixed energy level it is shown that as Φ decreases through 90° the two

degs

homoclinic orbits bifurcate into two homoclinic orbits, a periodic orbit, and connecting orbits. These orbits can then be used to define a

R superscript 4

compact region in $\mathbb{R}^4$. Now treating the energy as a parameter value the trajectory of orbits passing through this compact region can be described using symbolic dynamics. In this case it is shown that a single periodic orbit bifurcates into three periodic orbits whose stable and unstable manifold intersect transversely.

1. Introduction

In Hamiltonian systems, the existence of transverse intersections between the stable and unstable manifolds of distinct periodic orbits gives rise to regions in which solutions to the system exhibit complex or pathological behavior. For Hamiltonian systems arising from a Hamiltonian function, H: $\mathbb{R}^4 \to \mathbb{R}$, Rod [7], and Churchill and Rod [2], [3] and [4] established methods for showing the existence of such transverse intersections. Essential to their methods is the fact that along solutions of a Hamiltonian system the Hamiltonian function (i.e., the total energy) is constant. Thus, by considering a fixed energy level, they need only consider a 3-dimensional system. In the examples considered in [3], [4], and [7], the analysis is almost exclusively limited to the energy levels at which the complex behavior is exhibited.

In this paper, we treat the energy level as a bifurcation parameter, and present an example in which a single periodic orbit bifurcates into three periodic orbits whose stable and unstable sets intersect transversely. Our example differs from theirs in two other ways. First, we do not assume the existence of any global symmetries (compare with [3], [4], and [7]). For the sake of clarity, we introduce a local symmetry at the origin. However, this symmetry is not necessary and we sketch how the results can be established for the more general nonsymmetric case. Second, the Hamiltonian functions considered in [3], [4], [7], and here are all of the form $H(x,y) = \frac{1}{2}\langle y,y\rangle + V(x)$ where $x,y \in \mathbb{R}^2$. In the former papers, the origin is either a local minimum of V or a degenerate saddle point (i.e. there exist directions in which V increases and other directions in which V decreases). In our example, the origin is taken to be a non-degenerate local maxima of V, i.e.

$$D^2V(0) = \begin{bmatrix} a_1^2 & 0 \\ 0 & a_2^2 \end{bmatrix} .$$

As will be seen, the set of pathological bounded orbits lies arbitrarily close to the origin.

The energy level, $H = 0$, is the value at which the bifurcation takes place. It is assumed that on this energy level two orbits, homoclinic to the origin, appear. Furthermore, as the energy, H, is decreased, these homoclinic orbits become disjoint periodic orbits. Section 3 will show that if the homoclinic orbits approach the origin at a certain angle, then there exists another distinct periodic orbit. These three periodic orbits are the "basic" periodic orbits, whose stable and unstable sets are shown to intersect transversely. If we take the "angle" between the homoclinic orbits to be a bifurcation parameter (we are now letting the Hamiltonian function vary), then we have another bifurcation occuring. In this case, two orbits homoclinic to the origin, with their "nested" periodic orbits, bifurcate into a periodic orbit and a region exhibiting the above-mentioned pathological behavior.

Because we assume that the origin is a non-degenerate critical point of V (and hence H), the solutions to Hamilton's equations

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= -D_iV(x) \end{aligned} \tag{1.1}$$

can be approximated near the origin by solutions to the linear problem defined by $H_L(x,y) = \frac{1}{2}\langle y,y\rangle - \frac{1}{2}(a_1^2x_1^2 + a_2^2x_2^2)$. This is done in Section 2. In fact, in order to present the concepts clearly, we only consider the linear problem in Section 2.1. To understand how the solutions behave near the origin we restrict our attention to the set $\{z \mid H(z) = 0\}$ and replace the origin by a torus. This new space is called the critical manifold. On this critical manifold we define a new flow, compatible with that defined by (1.1), and use this new flow to analyze the behavior of solutions passing near the origin.

Whereas the results of Section 2 are local in nature, Section 3 begins the analysis of the global sructure of the solutions. As such we need to introduce the global non-linearities of V. These are given as a series of assumptions concerning the qualitative behavior of the flow generated by the Hamiltonian system, rather than explicit restrictions on the potential functions. There are reasons for choosing this indirect approach. First, the hypotheses of the theorems are qualitative in nature. Unfortunately, the analysis required to check that a particular potential function satisfies such hypotheses is often long, sometimes difficult, and usually ad hoc. For a discussion on the types of functions which give rise to this qualitative behavior, or on the types of arguments which can be employed to demonstrate such behavior, the reader is referred to [1], [2], [3], [4], and [7]. Second, in order to obtain pathologies in the manner described in this paper, one needs rather mild assumptions. In fact, most of the assumptions we make can be changed without significantly altering the results. We chose the conditions on V so as to emphasize the underlying causes of the results rather than to obtain the most general or most easily applicable results.

Also, in Section 3, we restrict our attention, for the most part, to

$$V(x_1,x_2) = -\frac{a^2}{2}(x_1^2 + x_2^2) + V_0(x) ,$$

where $V_0(x)$ is $O(\|x\|^2)$ at the origin. We use the results of Section 2 to prove the existence of an isolated periodic orbit which persists for all energy levels near the bifurcation point $H = 0$. Changing to the case where the angle between the homoclinic points is used as a bifurcation parameter, we prove (Theorem 3.20) that a periodic orbit bifurcates out of two homoclinic orbits when the angle is 90°. Finally, we comment briefly on how similar results could be obtained for the case $V(x_1,x_2) = \frac{1}{2}(a_1^2x_1^2 + a_2^2x_2^2) + V_0(x)$.

The results of Sections 4 and 5 depend heavily on the work of Rod [7] and Churchill and Rod [2]. Chapter 4 shows that the hypotheses of their theorems are satisfied. Unfortunately, developing the language in which to state the hypotheses is a lengthy process. Thus, rather than repeat a substantial portion of their papers [2], [3], and [7], it is assumed that the reader is familiar with their work, and hence, only the results which differ substantially from theirs are proved.

In Chapter 5, using symbolic dynamics, we classify the set of orbits which intersect a compact region defined by the "basic" periodic and homoclinic orbits near the origin. Our presentation of these results is very curt and the reader is referred to [2], [6] or [7] for a more complete interpretation of results of this type.

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2. The Critical Manifold

2.1. The Linear System

For the sake of clarity of exposition, we begin by considering a linear Hamiltonian system. For our purposes, the most general potential function we can choose is of the form, $-(a_1^2x_1^2 + a_2^2x_2^2)$. However, since we are only concerned with the qualitative behavior, we can scale out one of the coefficients to get,

$$V(x) = V(x_1,x_2) = -\frac{1}{2}(x_1^2 + a^2x_2^2), \quad a \geq 1. \tag{2.1}$$

This gives rise to the Hamiltonian function, $H: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ where

$$H(x,y) = \frac{1}{2}\langle y,y\rangle + V(x). \tag{2.2}$$

If we let $z = (x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$, then Hamilton's equations applied to (2.2) give the linear system of differential equations,

$$\dot{z} = Ax, \qquad \dot{} = d/dt \tag{2.3}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \end{bmatrix}.$$

By the chain rule we have that H is constant on solutions of (2.3). Let $M = \{z \in \mathbb{R}^4 \mid H(z) = 0\}$. One easily checks that the origin, O, is the only critical point of (2.3) and that $O \in M$. In order to understand the behavior of solutions of (2.3) which lie in M and pass near O, we will replace the critical point by a critical manifold and define a flow on our new space which corresponds to the old flow on this manifold. This section details the construction of the critical manifold for (2.3). As will be seen in the next section, this construction carries over to the class of non-linear Hamiltonian systems which interest us.

Since (2.3) is a linear system it is possible to rewrite the differential equations in polar form. Let $S^3(r) = \{z \in R^4 \mid \|z\| = 1\}$ and let $\zeta \in S^3(1)$. Given $z \in R^4\backslash\{O\}$ there exists a unique $r > 0$ and a unique ζ such that $z = r\zeta$. The polar form differential equations are given in the following lemma.

Lemma 2.1: *(a)* $\frac{d}{dt}<\zeta,\zeta> = 2<\dot{\zeta},\zeta> = 0$

(b) $\dot{r} = r<A\zeta,\zeta>$ (2.4)

(c) $\dot{\zeta} = A\zeta - <A\zeta,\zeta>\zeta$ (2.5)

Proof: (a) This follows from $<\zeta,\zeta> = 1$.

(b) Differentiating $z = r\zeta$ gives $\dot{z} = \dot{r}\zeta + rA\zeta$. Taking the inner product with ζ and using (a) gives (b).

(c) Substitute (b) into (c). □

Understanding the flow given by (2.4) and (2.5) turns out to be of great importance. Of particular interest is the fact that (2.5) is independent of r. This implies that $S^3(r)$ is an invariant set for (2.5), for all $r > 0$, which is not surprising since (2.5) is nothing more than the projection of the flow of (2.2) onto the unit sphere centered at the origin. Since $S^3(r)$ is an invariant set of (2.5) it makes sense to ask how the solutions restricted to $S^3(r) \cap M$ behave. Let P_1: $M \to \mathbb{R}^2$ be given by $P_1(x,y) = x$. Let $T(r) = S^3(r) \cap M$ and define $K(r) = \{x \mid 2x_1^2 + (1 + a)x_2^2 = r\}$.

Proposition 2.2: $T(r)$ *is homeomorphic to a torus, i.e.,* $S^1 \times S^1$.

Proof: $K(r)$ defines an ellipse since $r > 0$, and hence is homeomorphic to a circle, i.e., S^1. Given $x \in K(r)$ one can check that

$$P_1^{-1}(x) = \{(x,y) \mid \langle y,y\rangle = \frac{1}{2}(r + (a^2 - 1)x_2^2)\}.$$

This implies that for all $x \in K(r)$, $P_1^{-1}(x)$ is homeomorphic to S^1. We can think of $T(r)$ as a fiber bundle with base $K(r) \sim S^1$ and fiber $P_1^{-1}(x) \sim S^1$. Since y corresponds to the velocity vector it is clear that $T(r)$ is orientable. Thus $T(r)$ is a torus. □

The eigenvalues of A are ± 1 and $\pm a$. The corresponding eigenvector spaces are generated by $(x_1,0,x_1,0)$, $(x_1,0,-x_1,0)$, $(0,x_2,0,ax_2)$, and $(0,x_2,0,-ax_2)$. The stable manifold to the origin, W^s, is spanned by $(x_1,0,-x_1,0)$ and $(0,x_2,0,-ax_2)$. The unstable manifold, W^u, is spanned by $(x_1,0,x_1,0)$ and $(0,x_2,0,ax_2)$. Thus both W^u and W^s are two dimensional manifolds contained in M. Let

$$S^u(r) = W^u \cap S^3(r) \quad \text{and} \quad S^s(r) = W^s \cap S^3(r).$$

The following is obvious.

Lemma 2.3: *(a)* $S^u(r) \subset T(r)$ *and* $S^s(r) \subset T(r)$ *for all* $r \geq 0$.

(b) $S^u(r)$ *and* $S^s(r)$ *are homeomorphic to* S^1.

Since (2.5) is independent of r, the description of the flow on $T(r)$ will be the same regardless of what value of $r \geq 0$ is chosen. Hence, without loss of generality, one can, for purposes of simplifying the calculations let $r = 1$. To simplify the notation let $T = T(1)$, $S^3 = S^3(1)$, $S^u = S^u(1)$, and $S^s = S^s(1)$.

Lemma 2.4: *Solutions to* (2.5) *have the following properties:*

(a) If ζ *is an eigenvector of* A *then* $\dot{\zeta} = 0$.

(b) If $\zeta \in T$ *and* $\zeta \notin S^u \cup S^s$, *then* $\dot{\zeta} \neq 0$.

(c) Solutions on $T \backslash \{S^u \cup S^s\}$ *are heteroclinic orbits from fixed points of* S^s *to fixed points of* S^u.

If $a > 1$, *then the fixed points on* T *are the eigenvectors of* A *which lie on* T. *Furthermore, the flow on* S^s *consists of the four critical points* $\pm 2^{-1/2}(1,0,-1,0)$ *and* $\pm(1+a^2)^{-1/2}(0,1,0,-a)$ *plus heteroclinic orbits from* $(1+a^2)^{-1/2}(0,1,0,-a)$ *to* $\pm 2^{-1/2}(1,0,-1,0)$ *and from* $\pm(1+a^2)^{-1/2}(0,-1,0,a)$ *to* $\pm 2^{-1/2}(1,0,-1,0)$. *The flow on* S^u *consists of the critical points* $\pm 2^{-1/2}(1,0,1,0)$ *and* $\pm(1+a^2)^{-1/2}(0,1,0,a)$ *plus heteroclinic orbits from* $2^{-1/2}(1,0,1,0)$ *to* $\pm(1+a^2)^{-1/2}(0,1,0,a)$ *and from* $2^{-1/2}(-1,0,-1,0)$ *to* $\pm(1+a^2)^{-1/2}(0,1,0,a)$.

If $a = 1$ *then all the elements of* $S^s \cup S^u$ *are fixed points.*

Proof: All the results are evident if one recalls that (2.5) is the projection of the linear flow (2.3) onto the unit sphere. □

Since (2.5) is derived from a linear system, one might hope to be able to find a simple exact description of the heteroclinic orbits on T. If one assumes that $a = 1$ then this is the case. For $a > 1$ we shall not attempt to do so except for a few special orbits.

Notation 2.5: From now on ξ will denote a critical point in S^s and η a critical point in S^u. For $a > 1$, the possible values of ξ and η are given in Lemma 2.4. If $a = 1$ then $\xi = (\xi_1,\xi_2,-\xi_1,-\xi_2)$ and $\eta = (\eta_1,\eta_2,\eta_1,\eta_2)$.

Definition 2.6: For fixed ξ and η define a path in S^3 by

$$w(\xi,\eta) = w\colon\ [0,1] \to S^3$$

where

$$w(\xi,\eta;c) = \frac{(1-c)\xi + c\eta}{\|(1-c)\xi + c\eta\|} .$$

For fixed ξ and η, define $\Gamma(c) = \|(1-c)\xi + c\eta\|^{-1}$ and let $W(c) = \Gamma^{-1}(c)w(c)$. Finally, let $w(c) = (w_1(c),w_2(c),w_3(c),w_4(c))$.

Proposition 2.7: $w(\xi,\eta)\colon [0,1] \to T$ *if and only if* $\eta = \pm(\xi_2,-\xi_1, -\frac{1}{a}\xi_4,a\xi_3)$.

Proof: One needs to find the conditions on ξ and η such that for all $c \in [0,1]$, $H(w(c)) = 0$. By (2.2) this is the same as requiring that

$$w_3^2(c) + w_4^2(c) - w_1^2(c) - a^2 w_2^2(c) = 0. \tag{2.6}$$

At this point there is a multitude of cases which need to be checked. If $a > 1$ then the results follow by simple substitution. We shall demonstrate the case in which $a = 1$.

Substitution of Notation 2.5 and Definition 2.6 into equation (2.6) plus some simple calculations yield

$$\xi_1 \eta_1 = -\xi_2 \eta_2 \tag{2.7}$$

or

$$\xi_1^2 \eta_1^2 = \xi_2^2 \eta_2^2. \tag{2.8}$$

Since $\xi, \eta \in S^3$, one has that

$$2(\xi_1^2 + \xi_2^2) = 1 = 2(\eta_1^2 + \eta_2^2). \tag{2.9}$$

Using (2.9) to solve for ξ_1^2 and η_2^2 and substituting into (2.8) gives $\eta_1 = \pm\xi_2$. A similar argument gives $\eta_2 = \pm\xi_1$. The desired result now follows from (2.7) and (2.8). □

Definition 2.8: Given ξ define $\eta_+ = (\xi_2, -\xi_1, -\frac{1}{a}\xi_4, a\xi_3)$ and $\eta_- = (-\xi_2, \xi_1, \frac{1}{a}\xi_4, -a\xi_3)$.

Proposition 2.9: *The curves* $w(\xi,\eta_+;c)$ *and* $w(\xi,\eta_-;c)$, *for* $c \in (0,1)$, *represent heteroclinic orbits on* $T \backslash \{S^s \cup S^u\}$. *If* $a = 1$ *these are all the heteroclinic orbits described in Lemma* 2.4.

Proof: Again we are faced with a multitude of cases. We shall give the proof for $a = 1$ and η_+. The other cases follow in a similar manner. Let V be the vector space spanned by ξ and η_+. $w(c)$ is a curve which lies in $V \cap S^3$. If for any fixed c, $\zeta = w(c)$ and $\zeta \in V$, then $w(c)$ will represent a heteroclinic orbit. Equation (2.5) gives

$$\begin{aligned}\zeta &= Aw(c) - \langle Aw(c),w(c)\rangle w(c)\\ &= \Gamma AW(c) - \Gamma^3\langle AW(c),W(c)\rangle W(c).\end{aligned}$$

What needs to be shown is that there exist real numbers d and e such that $d\xi + e\eta_+ = \zeta$. Simple but tedious calculations give that:

$$AW = (c\xi_2 - (1-c)\xi_1, -(1-c)\xi_2 - c\xi_1, (1-c)\xi_1 + c\xi, (1-c)\xi_2 - c\xi_1),$$

$$\langle AW,W\rangle = 2c - 1,$$

$$d = -\Gamma(1-c)[1 + (2c-1)\Gamma^2],$$

and

$$e = \Gamma\, c\, [1 - (2c-1)\Gamma^2].$$

The details of checking that $d\xi + e\eta_+ = \zeta$ is satisfied is left to the reader. □

Consider (2.4) restricted to T. Notice that $z = r\zeta \in S^s$ implies $\dot r < 0$ and $z \in S^u$ implies $\dot r > 0$. We want to describe the set of points on $S^3(r)$ at which $\dot r = 0$. For $r > 0$ this means solving $\langle A\zeta,\zeta\rangle = 0$. One easily checks that if z lies in the vector space spanned by

$$\{(1,\ \sqrt{2}(a(1+a^2))^{-1/2},\ 1,\ -\sqrt{2a}(1+a^2)^{-1/2}),\ (1,\ \sqrt{2}(a(1+a^2))^{-1/2},\ -1,\ \sqrt{2a}(1+a^2)^{-1/2})\}$$

or, by

$$\{(-1,\ \sqrt{2}(a(1+a^2))^{-1/2},\ 1,\ -\sqrt{2a}(1+a^2)^{-1/2}),\ (-1,\ \sqrt{2}(a(1+a^2))^{-1/2},\ 1,\ \sqrt{2a}(1+a^2)^{-1/2})\}.$$

then $\langle Az,z\rangle = 0$, i.e., if $z = r\zeta$ then $r = 0$. On the other hand, since $\nabla_z\langle Az,z\rangle \neq 0$ for all $z \neq 0$, these planes are the only points for which $r = 0$. These two planes intersect $T(r)$ in two disjoint circles. Figure 1 describes the flow of (2.5) restricted to $T(r)$ for $a > 1$. If $a = 1$ then $r = 0$ at $w(\xi,\eta;c)$ if and only if $c = 1/2$. If $c \in [0,1/2)$, then $r < 0$ and if $c \in (1/2,1]$ then $r > 0$.

INSERT FIGURE 1.

So far, the results of this section have been independent of r, the only restriction being that $r > 0$. Now consider the case $r = 0$. In M, $r = 0$ corresponds to the origin which, according to (2.3), is a rest point. While (2.5) is still applicable when $r = 0$, it is of limited use when applied to a single point. Thus, to fully exploit (2.5) it is necessary to construct a critical manifold, CM, to replace M. In particular the origin in M, O, will be replaced by a torus, CT, on which (2.5) is defined in a non-trivial manner. The details are what follows.

Let $A \amalg B$ denote the disjoint union of two sets A and B.

Definition 2.10: Let $X = \mathbb{R}^4\backslash\{O\} \amalg S$. Define $h: X \to \{v \in \mathbb{R}^4 \mid \|v\| \geq 1\}$ by

$$h(z) = h(r\zeta) = \begin{cases} (r+1)\zeta & \text{if } z \in \mathbb{R}^4\backslash\{O\} \\ \zeta & \text{if } z \in S^3. \end{cases}$$

The topology of X is such that h is a homeomorphism.

The following system of coordinates will be used to describe the elements of X. If $z \in X$ then $z \in \mathbb{R}^4\backslash\{O\}$ or $z \in S^3$. In the first case, one writes $z = r\zeta = (r,\zeta)$. In the latter case, one writes $z = (0,\zeta)$. Using this notation, one can check that (2.4) and (2.5) are well defined on X. Furthermore, (2.4) and (2.5) give rise to a continuous flow on X.

Since $M\backslash\{O\} \subset \mathbb{R}^4\backslash\{O\}$ there is an obvious embedding of $M\backslash\{O\}$ into X given by $z = r\zeta \longmapsto (r,\zeta)$. In addition, under this embedding $M\backslash\{O\}$ is not closed in X. Define CM to be the closure of $M\backslash\{O\}$ in X. Let

$$CT = CM\backslash(M\backslash\{O\}).$$

Proposition 2.11: CT *is homeomorphic to a torus.*

Proof: Notice that $CT \subset S^3 \subset X$, hence $z \in CT$ implies that $z = (0,\zeta)$. Let $\zeta \in T$ then $(r,\zeta) \in M\backslash\{O\}$ for all $r > 0$. Thus

$$\lim_{r \to 0} (r,\zeta) = (0,\zeta) \in CT.$$

If $\zeta \in S^3\backslash T$ then $(0,\zeta)$ is not a limit point of $M\backslash\{O\}$ in X, thus $(0,\zeta) \notin CT$. Therefore CT and T are homeomorphic which by Proposition 2.2 implies that CT is homeomorphic to a torus. $\square$

The flow on CM is determined by (2.4) and (2.5). Notice that for (r,ζ), $r > 0$, one has the same flow as that determined by (2.3). However, on CT the flow arises from (2.4) and (2.5) for $r = 0$. Since elements of CT are of the form $(0,\zeta)$, CT is an invariant set of the flow. The flow is continuous on CM since (2.5) is independent of r and (2.4) is continuous in r for all $r \geq 0$. This flow, which is a mapping $CM \times \mathbb{R} \to CM$, will be denoted by

$$((r,\zeta),t) \longmapsto (r,\zeta)\cdot t = (r\cdot t, \zeta\cdot t)$$

where $r\cdot t$ is determined by (2.4) and $(\zeta\cdot t)$ is determined by (2.5).

Definition 2.12: For fixed $r_0 > 0$ define:

$$B(r_0) = \{(r,\zeta) \in CM \mid r \leq r_0\}$$

$$B^s(r_0) = \{(r_0,\zeta) \in CM \mid \dot{r} \leq 0\}$$

$$B^u(r_0) = \{(r_0,\zeta) \in CM \mid \dot{r} \geq 0\}.$$

Remarks 2.13: (a) $Tr(r) = B^s(r) \cup B^u(r)$.

(b) It follows from Proposition 2.2, Lemmas 2.3 and 2.4, and (2.5) that $B^s(r)$ and $B^u(r)$ are homeomorphic to annuli.

(c) $B^s(r) \cap B^u(r)$ is homeomorphic to two disjoint circles (see comments following Proposition 2.9) which will be denoted by $C_1(r)$ and $C_2(r)$.

(d) In the language of Conley [5], $B(r)$, $r > 0$ is an isolating neighborhood for the maximal invariant set CT. The exit set and entrance set for $B(r)$ are given by $B^u(r)$ and $B^s(r)$, respectively.

Proposition 2.14: *Given* $(r,\zeta) \in B^s(r)\backslash S^s(r)$ *there exists a unique* $t^* = t^*(r,\zeta) \geq 0$ *such that* $(r,\zeta)\cdot t^* \in B^u(r)\backslash S^u(r)$ *and* $(r,\zeta)\cdot[0,t^*] \subset B(r)$. *Define* φ: $B^s(r)\backslash S^s(r) \rightarrow B^u(r)\backslash S^u(r)$ *by* $\varphi(r,\zeta) = (r,\zeta)\cdot t^*(r,\zeta)$, *then* φ *is a homeomorphism.*

Proof: Because the flow defined on CM corresponds to the flow of the linear equations (2.3) for all elements except those on CT, it is clear that if $(r,\zeta)\cdot\mathbb{R} \subset B(r)$ then $(r,\zeta) \in T(0)$. Also, from the linearity of (2.3), one has that if $(r,\zeta)\cdot t \in B(r)$ for all $t \geq 0$ then $(r,\zeta) \in S^s(r)$. Likewise, if $(r,\zeta)\cdot t \in B(r)$ for all $t < 0$ then $(r,\zeta) \in S^u(r)$. Thus, one has the existence of t^* if $(r,\zeta) \in B^s(r)\backslash S^s(r)$, and one has that

$$\varphi(B^s(r)\backslash S^s(r)) = B^u(r)\backslash S^u(r).$$

That φ is a homeomorphism follows from the uniqueness of solutions of ordinary differential equations. □

Corollary 2.15: φ *is the identity on* $C_1(r) \cup C_2(r) = B^s(r) \cap B^u(r)$.

In what follows, it will be useful to keep in mind that $M \subset CM$. Thus, if $(r,\zeta) \in CM$ and $r > 0$ then

$$P_1(r,\zeta) = (r\zeta_1, r\zeta_2) \in \mathbb{R}^2.$$

Also, in most cases, when one deals with the sets $B(r)$, $B^u(r)$, and $B^s(r)$ it is assumed that $r > 0$ and fixed. Hence, for convenience sake, let $B = B(r)$, $B^u = B^u(r)$, and $B^s = B^s(r)$.

Remarks 2.16: Recall that the purpose of the critical manifold is to describe the behavior of solutions of (2.3) passing near the origin. This will be done by describing the map φ using Figures 2, 3, and 5, and the fact that the following conventions have been adopted.

(a) The projection under P_1 of a radial line in Figures 2, 3 and 5 is always a single point x. In the language of the proof of Proposition 2.2, the radial lines are subsets of the fibers.

(b) The projection under P_1 of a concentric circle is K(r).

(c) Let $L_i \subset \mathbb{R}^2$, $i = 1,2$ be rays emanating from the origin with slope m and $-m$, $m > 0$. Let L_1 lie in the first quadrant, and L_2 lie in the fourth quadrant, and let Ψ denote the angle between the line segments. Let $\Lambda_i = P_1^{-1}(L_i)$ for $i = 1,2$. Define $\Lambda_i^u = \Lambda_i \cap B^u$ and $\Lambda_i^s = \Lambda_i \cap B^s$. Then $\Lambda_i^s \backslash S^s$ consists of two line segments. Denote these line segments by $\Lambda_{i,1}^s$ or $\Lambda_{i,2}^s$ depending on whether they intersect C_1 or C_2, respectively. Similar definitions can be made for $\Lambda_{i,j}^u$, $j = 1,2$. Finally, define $d_j^u = C_j \cap \Lambda_{1,j}^u$, $d_j^s = C_j \cap \Lambda_{1,j}^s$, $e_j^u = C_j \cap \Lambda_{2,j}^u$ and $e_j^s = C_j \cap \Lambda_{2,j}^s$.

(d) Let $(x,y) = d_2^u$ where $y = (ny_1,-y_1)$.

(e) Let J represent the convex region in $\mathbb{R}^2$ bounded by L_i, $i = 1,2$. The shaded regions in Figure 2 consist of those elements of B which project under P_1 into J.

INSERT FIGURE 2.

INSERT FIGURE 3.

For the moment we restrict our attention to the case $a = 1$. A typical element of $S^s(r)$ must be of the form (r,ξ). Furthermore, $\lim_{t\to\infty} (r,\xi)\cdot t = (0,\xi) \in S^s(0) \subset CT$. By Proposition 2.9 and Definition 2.8, $w(\xi,\eta_\pm;c)$ describes the two heteroclinic orbits on CT connecting $(0,\xi) \in S^s(0)$ to $(0,\eta_\pm) \in S^u(0)$. In addition, $\lim_{t\to-\infty}(r,\eta_\pm) = (0,\eta_\pm)$. This allows us to define the following paths in CM.

Definition 2.17: Given any $(r,\xi) \in S^s$ let $\gamma_\pm(\xi)\colon [0,3] \to B(r)$ denote the two paths defined by

$$
\begin{aligned}
&\gamma_\pm(\xi)(s) = (r,\xi) \cdot s/1-s && \text{for } s \in [0,1)\\
&\gamma_\pm(\xi)(1) = (0,\xi) &&\\
&\gamma_\pm(\xi)(s) = (0,w(\xi,\eta_\pm;\, s-1)) && \text{for } s \in (1,2)\\
&\gamma_\pm(\xi)(2) = (0,\eta_\pm) &&\\
&\gamma_\pm(\xi)(s) = (r,\eta_\pm) \cdot (3-s)/(2-s) && \text{for } s \in (2,3].
\end{aligned}
$$

Definition 2.18: Let d be a metric on CM. Let $\gamma\colon [0,3] \to CM$. Let $(r,\zeta) \in CM$. One says that the orbit of (r,ζ) lies in an *ϵ-tube about γ over the interval* $[t_0,t_1]$ if for all $t \in [t_0,t_1]$

$$\inf_{s\in[0,3]} \{d(\gamma(s),(r,\zeta)\cdot t\} < \epsilon.$$

Proposition 2.19: *If* $a = 1$ *then the images of* $\Lambda^s_{i,j}$ for $i = 1,2,\ j = 1,2,$ *under* φ *are as given in Figure* 3. *(Notice the importance of* Ψ*).*

Proof: Corollary 2.15 fixed the elements of C_1 and C_2 under φ.

Let $(r,\zeta) \in B^*\backslash S^*$, then by Proposition 2.14 there exists a unique $t^* = t^*(r,\zeta)$ such that $(r,\zeta)\cdot[0,t^*] \subset B$. Let, $(r,\zeta) \in S^*(r)$. Recall the construction of $\gamma_\pm(\xi)$. Each "piece" of $\gamma_\pm$ is made up of a solution to (2.4) and (2.5). Thus, by continuity one can choose $(r,\zeta) \in B^*\backslash S^*$ such that the orbit of (r,ζ) lies in an ϵ-tube of either $\gamma_+(\xi)$ or $\gamma_-(\xi)$ over the interval $[0,t^*]$. Remarks 2.16 (d) and (e) force $d_2^* = (r,\zeta_1,m\zeta_1,m\zeta_3,-\zeta_3)$ and hence $P_1(\Lambda_{1,2}^*\cdot t) \notin J$ for all $t > 0$.

Finally, notice that $\langle P_1(r,\eta_+),P_1(r,\xi)\rangle = 0$ and $P_1(r,\eta_+) = -P_1(r,\eta_-)$. Thus the image of $\Lambda_{1,2}^*$ under φ must be as shown. The arguments for $\varphi(\Lambda_{i,j}^*)$ are similar. $\square$

A technically incorrect but intuitively illuminating restatement of Proposition 2.19 is as follows. If one considers the path of a solution to (2.3) in x-space, then those orbits which lie on the stable manifold to the origin, leave the origin on the unstable manifold in a direction perpendicular to the direction of entry. In application, this means that orbits on the surface M which pass close to the origin change direction by slightly less than 90°. See Figure 4.

INSERT FIGURE 4.

We now consider briefly the case $a > 1$. Let

$$\pm\xi^1 = \pm\left[0, \frac{1}{\sqrt{1+a^2}}, 0, \frac{-a}{\sqrt{1+a^2}}\right],$$

$$\pm\xi^2 = \pm\left[\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}, 0\right], \quad \pm\eta^1 = \pm\left[0, \frac{1}{\sqrt{1+a^2}}, 0, \frac{a}{\sqrt{1+a^2}}\right]$$

and $\pm\eta^2 = \pm(1/\sqrt{2},\ 0,\ 1/\sqrt{2},\ 0)$. Let $(r,\zeta) \in S^3(r)$. If $\zeta \neq \pm\xi^1$ then $\lim_{t\to\infty} (r,\eta)\cdot t = (0,\pm\xi^2)$. Assume $\lim_{t\to\infty} (r,\zeta)\cdot t = (0,\xi^2)$. Then we can define $\gamma_\pm(\zeta)\colon [0,\xi] \to B(r)$ as follows:

$$\begin{aligned}
&\gamma_\pm(\zeta)(s) = (r,\zeta)\cdot s/1-s && \text{for } s \in [0,1)\\
&\gamma_\pm(\zeta)(1) = (0,\xi^2) && \\
&\gamma_\pm(\zeta)(s) = (0,\ w(\xi^2,\eta^1_\pm;\ s-1)) && \text{for } s \in (1,2)\\
&\gamma_\pm(1g)(2) = (0,\eta^1_\pm) && \\
&\gamma_\pm(\zeta)(s) = (r,\eta^1_\pm)\cdot 3-s/(2-s) && \text{for } s \in (2,3].
\end{aligned}$$

If $\lim_{t\to\infty} (r,\zeta)\cdot t = (0,-\xi^2)$ then there is a corresponding definition for $\gamma_\pm(\zeta)$. The question of how to define $\gamma_\pm(\pm\xi^1)$ is more delicate. Because the system is linear, we have an exact solution for (2.3), namely, $z(t) = e^{At}z_0$. Using this, we conclude that $\gamma_\pm$ should satisfy

$$\gamma_\pm(\xi^1)(2) = (0,\eta^2_\pm) = \gamma_\mp(-\xi^1)(2).$$

(This involves checking that $(r,\zeta) \in CM$ close to (r,ξ^1) implies that $(r,\zeta)\cdot t^*(r,\zeta)$ is not close to $(r,\eta^1_\pm)$). A proof similar to that for Proposition 2.19 says that Figure 5 demonstrates how φ acts on $B^s\backslash S^s$.

INSERT FIGURE 5.

2.2. The Nonlinear System

The results of Section 2.1 are easily extendable to the class of non-linear Hamiltonian functions, $H \in C^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$, given by

$$H(x,y) = \frac{1}{2} \langle y,y \rangle + V(x) \tag{2.10}$$

where

$$V(x) = -\frac{1}{2} (a_1^2 x_1^2 + a_2^2 x_2^2) + V_0(x) \tag{2.11}$$

and $V_0(x) = o(\|x\|^2)$ at the origin. As before, we note that the qualitative picture near the origin will not change if we set $a_1 = 1$. The differential equations of interest are given by Hamilton's equation, i.e.,

$$\begin{aligned} x &= y \\ \dot{y} &= -\nabla V(x). \end{aligned} \tag{2.12}$$

As in the case of (2.3), the origin, O, is a fixed point for (2.12) and H (i.e., (2.10)) is constant along solutions. Thus $M = \{z \mid H(z) = 0\}$ is invariant under (2.12). As before we can embed $M \backslash \{O\}$ into X and define CM to be the closure of $M \backslash \{O\}$ in X. Define CT = $CM \backslash (M \backslash \{O\})$. The question is whether the flow on CT determined by (2.12) is the same as that of (2.3). The following theorem answers it in the affirmative.

Theorem 2.20: *(Hartman-Grobman). Let* $z = f(z)$, $z \in \mathbb{R}^n$ *and* $f(\bar{z}) = 0$. *If* $Df(\bar{z})$ *has no zero or purely imaginary eigenvalues then there is a homeomorphism* h *defined on some neighborhood* U *of* $\bar{z}$ *in* $\mathbb{R}^n$ *locally taking orbits of the nonlinear flow* φ_t, *corresponding to* $z = f(z)$, *to those of the linear flow* $e^{tDf(\bar{z})}$ *corresponding to* $z = Df(\bar{z})z$. *The homeomorphism preserves the sense of orbits and can also be chosen to preserve parametrization by time.*

3. Periodic Orbits

This section contains theorems concerning the existence of periodic orbits arising from the Hamiltonian systems of the form (2.12). It is worth emphasizing that these periodic orbits do not occur because of a global symmetry but rather because of the local behavior described in Section 2. For the sake of clarity in our exposition we shall assume

$$V(x) = -\frac{1}{2}(x_1^2 + x_2^2) + V_0(x).$$

This is a strong assumption as it introduces local symmetry. However, it is not crucial to the types of arguments used in the proofs that follow.

Let $E(h) = \{x \mid V(x) = h\}$, then E is called an *equipotential set* of V. Let $h^- < 0 < h^+$. h^- and h^+ will be lower and upper bounds for h and need to be chosen sufficiently small for the following results to hold. We do not attempt to estimate what these values should be.

Assumption 1:

(a) *If* $h \in (h^-,0)$ *then* $E(h) = EO(h) \cup ET(h) \cup EB(h)$ *where* $EO(h)$, $ET(h)$ *and* $EB(h)$ *are disconnected curves in* $\mathbb{R}^2$. *(See Figure 6.) If* $x = (x_1,x_2) \in ET(h)$ *then* $x_2 > 0$, *and if* $x \in EB(h)$ *then* $x_2 < 0$. *Furthermore* $ET(h)$ *and* $EB(h)$ *bound the regions* $\{x \mid V(x) > h\}$ *away from the* x_1*-axis.* $EO(h)$ *is the boundary of a region* $\{x \mid V(h) > h\}$ *which contains the origin.*

(b) If $h = 0$ *then* $E(h)$ *is as in (a). However,* $EO(h)$ *is the origin.*

(c) If $h \in (0,h^+)$ *then* $E(h)$ *is as in (a). However,* $EO(h)$ *is the empty set.*

INSERT FIGURE 6.

To simplify the notation, if $h = 0$ let $E = E(0)$, $ET = ET(0)$, and $EB = EB(0)$. Let $u: [0,1] \to ET$, $v: [0,1] \to EB$ be parametrizations of portions of ET and EB respectively. (See Figure 7.)

As before, the origin in $\mathbb{R}^4$ is a critical point with 2-dimensional stable and 2-dimensional unstable manifolds denoted by W^s and W^u, respectively. Let $P_i(h): \{z \mid H(z) = h\} \to \mathbb{R}^2$, $i = 1,2$, be given by $P_1(h)(x,y) = x$ and $P_2(h)(x,y) = y$. Again, to simplify the notation we write $P_i = P_i(0)$. It is easily checked that, if $H(x,y) = h$ and $x \in E(h)$ then $y = 0$. Thus, no confusion should arise if one considers $E(h) \subset \mathbb{R}^2$ or $E(h) \subset \{(x,y) \mid H(x,y) = h\}$. In particular, $u(s)$ and $v(s)$ will be used interchangeably to denote elements of $\mathbb{R}^2$ and elements of M.

Definition 3.1: Given $z \in \mathbb{R}^4$ such that $H(z) = h$, define

$$\lambda(z) = \inf\{t > 0 \mid P_1(h)(z \cdot t) \text{ lies on the } x_1\text{-axis}\}.$$

Assumption 2:

(a) $u(0)$ *and* $v(0)$ *lie on the stable manifold of the origin.*

(b) $\lambda(u(s))$ *and* $\lambda(v(s))$ *exist for all* $s \in (0,1]$.

(c) $P_1(u(s) \cdot \lambda(u(s)) = P_1(v(s) \cdot \lambda(v(s)))$ *for all* $s \in (0,1]$. *(See Figure 7.)*

INSERT FIGURE 7.

Let $P_i^*: X \to [-1,1] \times [-1,1]$ for $i = 1,2$ where $P_1^*(r,\zeta) = (\zeta_1,\zeta_2)$ and $P_2^*(r,\zeta) = (\zeta_3,\zeta_4)$. By assumption 2(a), $u(0)$ and $v(0)$ are elements of W^s. Thus $u(0)$ and $v(0)$ are elements of M. By the conventions of Section 2

$$\begin{aligned} &\lim_{t\to\infty} u(0) \cdot t = (0,\xi(u)) \in CT \\ &\lim_{t\to\infty} v(0) \cdot t = (0,\xi(v)) \in CT. \end{aligned} \tag{3.2}$$

Definition 3.2: Let Φ, $0° \leq \Phi \leq 180°$, be the angle defined by

$$\cos \Phi = \frac{P_1^*(\xi(u))}{\|P_1^*(\xi(u))\|} \cdot \frac{P_1^*(\xi(v))}{\|P_1^*(\xi(v))\|} .$$

Assumption 3: $0° < \Phi < 90°$.

Assumption 4: *If* $(x_1(t),x_2(t)) = P_1(u(0)\cdot t)$ *then* $x_1(t) > 0$ *and* $x_2(t) > 0$ *for all* $t \in [0,\infty)$. *If* $(x_1(t),x_2(t)) = P_1(v(0)\cdot t)$ *then* $x_1(t) > 0$ *and* $x_2(t) < 0$ *for all* $t \in [0,\infty)$.

Definition 3.3: Let $\alpha(s) \in [-\pi,0]$ and $\beta(s) \in [0,\pi]$ for $s \in (0,1]$ be defined by

$$\cos(\alpha(s)) = \frac{\langle P_2(u(s)\cdot\lambda(u(s))),(1,0)\rangle}{\|P_2(u(s)\cdot\lambda(u(s)))\|}$$

$$\cos(\beta(s)) = \frac{\langle P_2(v(s)\cdot\lambda(v(s))),(1,0)\rangle}{\|P_2(v(s)\cdot\lambda(v(s)))\|}$$

Geometrically, $\alpha(s)$ and $\beta(s)$ represent the angles through which the orbits originating at $u(s)$ and $v(s)$, respectively, cross the x_1-axis for the first time.

Definition 3.4: $\Psi(s) = \beta(s) - \alpha(s)$.

Assumption 5: $\Psi(1) < \pi$.

Theorem 3.5: *Given assumptions* 1-5, *there exists at least one periodic orbit on the energy surface* $\{z \mid H(z) = 0\}$.

The proof of this theorem is straightforward once one deals with the following two technicalities. First, at present Ψ is only defined for $s \in (0,1]$. One needs to extend the definitions of α and β in such a way that they are continuous functions on the closed interval $[0,1]$. This in turn will mean that Ψ is a continuous function on $[0,1]$. Second, one needs to know the value of $\Psi(0)$. As will be shown, Assumption 3 forces $\Psi(0) > \pi$. Assume these problems have been dealt with.

Proof: Since Ψ is continuous on $[0,1]$, $\Psi(0) > \pi$, and $\Psi(1) < \pi$, there exists an $s^* \in (0,1)$ such that $\Psi(s^*) = \pi$. This in turn implies that the orbit passing through $u(s^*)$ crosses the x_1-axis in exactly the opposite direction from the orbit passing through $v(s^*)$. Since $V(u(s^*)) = 0 = V(v(s^*))$, the velocity at these points is zero. Thus, the same orbit passes through $u(s^*)$ and $v(s^*)$. Invoking the reversibility of Hamiltonian systems, one has that the same orbit passes through $u(s^*)$ and $v(s^*)$ and hence that this orbit is periodic. □

The critical manifold can be used to define $\alpha(0)$ and $\beta(0)$ in such a way that α and β are continuous functions in $[0,1]$. For the moment, consider only the function α. As was mentioned before, $\alpha(s)$ represents the angle at which the orbit originating at $u(s)$ crosses the x_1-axis for the first time. By (3.2) $\lim_{t\to\infty} u(0)\cdot t = (0,\xi(u)) \in CT$. Thus we have the two curves $\gamma_\pm: [0,3] \to CM$ defined in Section 2 such that $\gamma_\pm(0) = u(0)$, $\gamma_\pm(1) = \xi(u)$ and $\gamma_\pm(2) = \eta_\pm(u)$.

Proposition 3.6: *For* s *sufficiently small, the orbit of* $u(s)$ *lies in an* ϵ*-tube about* γ_+ *over the interval* $[0,t^*]$.

Proof: By continuity of the flow, given any $\epsilon > 0$, there exists $t^* > 0$, such that for s sufficiently small the orbit $u(s)$ lies in an ϵ-tube about γ_+ or γ_- over the interval $[0,t^*]$. Assume the latter, i.e., $u(s)$ is close to γ_-. For all $s \in (0,1]$ one has that $P_1^*(u(s)\cdot\lambda(u(s))) = (x_1,0)$ where $x_1 > 0$. Hence one must be able to solve $P_1^*(w(\xi(u),\eta_-(u);c) = (x_1,0)$ where $x_1 > 0$. We shall show that this is not possible.

Let $\xi(u) = (p,mp,-p,-mp)$ where $p > 0$ and $m > 0$. That $\xi(u)$ must be of this form follows from Assumption 4 and Notation 2.5. This in turn implies $\eta_-(u) = (-mp,p,-mp,p)$. Thus, in order for $P_1^*(w(\xi(u),\eta_-(u);c) = (x_1,0)$, it must be the case that $(1-c)mp + cp = 0$ thus $c = m/m-1$. But $c \in (0,1)$ hence $m < 0$. Contradiction. □

A straightforward calculation gives:

Proposition 3.7: *If* $\xi(u) = (p,mp,-p,-mp)$ *where* $p > 0$ *and* $m > 0$ *and if* $P_1^*(w(\xi(u),\eta_+(u);c)) = (x_1,0)$ *where* $x_1 > 0$ *then* $c = m/m + 1$.

Let $\xi(v) = (q,-nq,-q,nq)$ where $q > 0$ and $n > 0$. Let $\gamma_-: [0,3] \to CM$ such that $\gamma_-(0) = v(0)$, $\gamma_-(1) = \xi(v)$ and $\gamma_-(2) = (nq,q,nq,q)$.

Proposition 3.8: *For* s *sufficiently small, the orbit of* v(s) *lies in an* ϵ-tube *about* γ_- *over the interval* $[0,t^*]$. *Furthermore, if* $P_1^*(w(\xi(v),\eta_-(v);c)) = (x_1,0)$, $x_1 > 0$ *then* $c = n/(n+1)$.

The proof of Proposition 3.8 is similar to that of Propositions 3.6 and 3.7. Direct calculation gives

$$P_1^*(w(\xi(u),\eta_+(u);\ m/m+1)) = \frac{1}{\sqrt{2}}\frac{1}{1+m^2}(m^2-1,\ -2m)$$

$$P_1^*(w(\xi(v),\eta_-(v);\ n/n+1)) = \frac{1}{\sqrt{2}}\frac{1}{1+n^2}(n^2-1,\ -2n).$$

Definition 3.9: $\alpha(0) \in [-\pi,0]$ and $\beta(0) \in [0,\pi]$ are given by

$$\cos(\alpha(0)) = \frac{P_2^* w}{\|P_2^* w\|} \cdot (1,0) = \frac{m^2-1}{m^2+1}$$

$$\cos(\beta(0)) = \frac{P_2^* w}{\|P_2^* w\|} \cdot (1,0) = \frac{n^2-1}{n^2+1}.$$

Using this definition one has that α and β are continuous on $[0,1]$ and hence, that Ψ is continuous on $[0,1]$.

Proposition 3.10: $\Psi(0) < \pi$.

Proof: Let

$$I = \frac{P_2^*(w(\xi(u),\eta_+(u);\ m/m+1))}{\|P_2^*(w(\xi(u),\eta_+(u);\ m/m+1))\|}$$

and

$$I^* = \frac{P_2^*(w(\xi(v),\eta_-(v);\ n/n+1))}{\|P_2^*(w(\xi(v),\eta_-(v);\ n/n+1))\|}.$$

Then $I = 1/(m^2+1)\ (m^2-1,-2m)$ and $I^* = 1/(n^2+1)\ (n^2-1,2n)$. Since $m,n > 0$ it is clear that I lies in quadrants III and IV of the plane, and I^* lies in quadrants I and II of the plane. If it can be shown that I^* lies to the left of $-I$ then clearly $\Psi(0) > \pi$. Let $(a,b) = -I - I^*$. Showing that Ψ lies to the left of $-I$ is equivalent to showing that $a > 0$.

$$a = (1-m^2)/(1+m^2) + (1-n^2)/(1+n^2)$$
$$= 2(1-m^2n^2)/(m^2+1)(n^2+1).$$

but Assumption 3 implies that $mn < 1$ and hence $a > 0$. □

Let us for the moment consider the case $a > 1$ in (2.12), and how it differs from what we have just done for $a = 1$. The x_1-axis has been singled out as a reference line in Assumptions 1, 2, and 4. When $a = 1$ this is not a restriction since V is locally symmetric about the origin. If $a > 1$ then the results one obtains will depend upon the reference line chosen. Assumption 2(a) works for $a = 1$ since all elements of S^s are critical points. If $a > 1$ then the results will change depending on whether $u(0)\cdot t$ converges to $\pm 1/\sqrt{2}(1,0,-1,0)$ or $\pm 1/\sqrt{1+a^2}(0,1,0,-a)$. Finally, we were able to give a sharp estimate for Assumption 3 (and hence Theorem 3.5) because we knew how the orbits $u(s)$ converged to $\gamma_\pm(u(s))$ as $s \to 0$. If $a > 1$ then the limit of the crossing angles will be sensitive to m, n, and a. Therefore, while one can perform the same type of analysis for $a > 1$, the arguments will have to be more delicate or the resulting theorems less precise.

Definition 3.11: Let $M(n) = \{z \mid H(z) = h\}$. Let $u(h)$: $[0,1] \to ET(h)$ and $v(h)$: $[0,1] \to EB(h)$ be parametrizations of portions of $ET(h)$ and $EB(h)$ respectively.

The results obtained up to this point have been proven only on the invariant surface M. This is due to the fact that any orbit which lies in $M(h)$, where $h \neq 0$, is bounded away from the origin in R^4.

Therefore, one cannot expect that the flow on $S(0)$ would provide a reasonable approximation to the orbits on $M(h)$. However, having found results for $M(0)$, slightly stronger conditions on V as well as the fact that V is continuous, should allow one to conclude that similar results hold for $M(h)$, as long as h is chosen sufficiently close to 0.

Assumption 6:

(a) u *and* v *are continuous on* $[h^-,h^+] \times [0,1]$.

(b) *For all* $s \in (0,1]$, $\lambda(u(h,s))$ *and* $\lambda(v(h,s))$ exist.

(c) $P_1(u(h,s) \cdot \lambda(u(h,s))) = P_1(v(h,s)) \cdot \lambda(v(h,s))$.

Definition 3.12: $\alpha(h,s) \in [-\pi,0]$ and $\beta(h,s) \in [0,\pi]$ are defined on $[h^-,h^+] \times (0,1]$ by

$$\cos(\alpha(h,s)) = \frac{\langle P_2(u(h,s) \cdot \lambda(u(h,s))),(1,0)\rangle}{\|P_2(u(h,s) \cdot \lambda(u(h,s)))\|}$$

$$\cos(\beta(h,s)) = \frac{\langle P_2(v(h,s) \cdot \lambda(v(h,s))),(1,0)\rangle}{\|P_2(v(h,s) \cdot \lambda(v(h,s)))\|}$$

$$\Upsilon(h,s) = \beta(h,s) - \alpha(h,s).$$

Notice that α, β and hence Υ, are continuous on $[h^-,h^+] \times (0,1]$.

Theorem 3.13: *Given Assumptions 1 - 6 and given* $|h^+|$ *and* $|h^-|$ *sufficiently small, there exists at least one periodic orbit,* $\Pi_3(h)$, *which lies in* $M(h)$ *and intersects* $ET(h)$ *and* $EB(h)$.

Proof: By Theorem 3.5 there exists an s^* such that $\Psi(0,s^*) = \pi$. Furthermore, by Proposition 3.10, $\Psi(0,0) > \pi$. Thus, there exists an $\tilde{s} \in (0,s^*)$ such that $\Psi(0,\tilde{s}) > \pi$. Since Ψ is continuous in a neighborhood of $(0,\tilde{s})$, there exists $\delta_0 > 0$ such that $\Psi(\epsilon,\tilde{s}) > \pi$ for $\epsilon \in [-\delta_0,\delta_0]$. Similarly, by Assumption 5 $\Psi(0,1) < \pi$, hence there exists $\delta_1 > 0$ such that $\Psi(\epsilon,1) < \pi$ for $\epsilon \in [-\delta_1,\delta_1]$.

Let $h^- = \max(-\delta_0,-\delta_1)$ and $h^+ = \min(\delta_0,\delta_1)$. Then $h \in [h^-,h^+]$ implies that there exists $s^*(h)$ such that $\Psi(h,s^*(h)) = \pi$. $\square$

From now on $h^\pm$ will be chosen as in Theorem 3.13.

Assumption 7: *If* $s_1 < s_2$ *then* $|\alpha(h,s_1)| > |\alpha(h,s_2)|$ *and* $|\beta(h,s_1)| > |\beta(h,s_2)|$.

Lemma 3.14: *Assumptions* 1-7 *insure that* $\Pi_s(h)$ *is unique for a given* $h \in [h^-,h^+]$.

Definition 3.15: Let $s_0(h) \in (0,\tilde{s}(h)]$ such that

$$u(s_0(h))\cdot[0,\lambda(u(s_0(h)))] \cap u(\tilde{s}(h))\cdot[0,\lambda(u(\tilde{s}(h)))] \neq \phi.$$

But, if $s \in (0,s_0(h))$ then

$$u(h,s)\cdot[0,\lambda(u(h,s))] \cap u(\tilde{s}(h))\cdot[0,\lambda(u(\tilde{s}(h)))] = \phi.$$

Let $x(h,s)$ be the x_1-coordinate of $P_1(u(h,s))\cdot\lambda(u(h,s))$. Let $s_1(h) \in (0,\tilde{s}(h)]$ such that $x(h,s_1(h)) = x(h,s^*(h)$ and if $s \in (0,s_1(h))$ then $x(h,s) < x(h,s(h))$.

Lemma 3.16: $x(h,s_0(h)) \leq x(h,\tilde{s}(h))$ *and if* $s < s_0$ *then* $x(h,s) < x(h,\tilde{s}(h))$.

Proposition 3.17: $s_0(h) = s_1(h) = \tilde{s}(h)$.

Proof: In what follows, h is considered fixed and hence, to simplify the notation, is suppressed. By definition $s_0 \leq \tilde{s}$. So assume $s_0 < \tilde{s}$. Furthermore assume $x(s_0) = x(\tilde{s})$. By Assumption 7 $|\alpha(s_0)| > |\alpha(\tilde{s})|$, hence one must be in the situation of Figure 8. But by continuity of the flow, there exists $s" < s_0$ such that

$$u(s") \cdot [0,\lambda(u(s"))] \cap u(\tilde{s}) \cdot [0,\lambda(u(\tilde{s}))] \neq \phi.$$

Contradiction. Thus $x(s_0) < x(\tilde{s})$. Now either $u(\tilde{s})\cdot[0,\lambda(u(s))]$ intersects $u(\tilde{s}\cdot[0,\lambda(u(\tilde{s}))]$ topologically transversally or not i.e., tangentially but not topologically transversally. The former cannot happen since this forces the existence of $s"$ as above. But if the intersection is tangential then

$$u(s_0) \cdot [0,\lambda(u(s))] = u(\tilde{s})\cdot[0,\lambda(u(\tilde{s}))].$$

Therefore, $s_0 = \tilde{s}$.

Thus either $0 < s_1 < s_0 = s^*$ or $s_1 = s_0 = s^*$. Assume the former then $x(s_1) < x(s^*)$. Contradiction. □

INSERT FIGURE 8.

Corollary 3.18: *Given Assumptions* 1-7:

(a) $u(h,s)\cdot[0,\lambda(u(h,s))] \cap u(h,S)\cdot[0,\lambda(u(h,S))] = \phi$ *if* $s \neq S$.

(b) $v(h,s)\cdot[0,\lambda(v(h,s))] \cap v(h,S)\cdot[0,\lambda(v(h,S))] = \phi$ *if* $s \neq S$.

Proof: The machinery developed starting with Lemma 3.14 proves (a). The proof for (b) is similar □

Theorem 3.19: *Given Assumptions* 1-7, *for* $h \in [h^-,h^+]$, $\Pi_3(h)$ *is an isolated periodic orbit in* $M(h)$.

Proof: First one constructs what will be the isolating set. Since $s^*(h)$ is the unique solution to $\Psi(s,h) = \pi$, it must be that for $0 < s_1 < s^* < s_2 < 1$, one has $\Psi(h,s_1(h)) > \pi > \Psi(h,s_2(h))$.

The orbits $P_1(u(s_i(h))\cdot[0,\lambda(u(s_i(h)))]$ and $P_1(v(s_i(h))\cdot[0,\lambda(v(s_i(h)))]$ for $i = 1,2$ can be used to define a compact region, denoted $N_{12}(h)$, contained in the set $\{x \mid V(x) \leq h\}$. (See Figure 9) Let

$$PN_{12}(h) = P_1^{-1}(h)(N_{12}(h)).$$

To see that $PN_{12}(h)$ is an isolating neighborhood of $\Pi_3(h)$, one must show that $PN_{12}(h)$ has no internal tangencies on $PN_{12}(h)$. By construction, the only tangential orbits are those which pass through $u(s_i(h))$ or $v(s_i(h))$ for $i = 1,2$. However, the conditions on $\alpha(s_i(h))$ or $\beta(s_i(h))$ force the orbits to leave in positive and negative time at the points

$$u(s_i(h)) \cdot \lambda(u(s_i(h)))$$

and

$$v(s_i(h)) \cdot \lambda(v(s_i(h))).$$

Thus, there are no internal tangencies.

One now needs to show that $\Pi_3(h)$ is the maximal invariant set in $PN_{12}(h)$. Assume not, i.e., assume that there exists some other orbit $\Pi' \neq \Pi_3(h)$ which is contained in $PN_{12}(h)$ for all time. Given $s_j(h)$ and $s_k(h)$ such that

$$s_1(h) \leq s_j(h) \leq s^*(h) \leq s_k(h) \leq s_2(h) \; ,$$

construct $PN_{jk}(h) \subset M(h)$ in the same manner that $PN_{12}(h)$ was constructed from s_1 and s_2. Let

$$PN(h) = \bigcap_{j,k} \{PN_{jk}(h) \mid \Pi' \subset PN_{jk}(h)\}.$$

By Corollary 3.18, there exists $s_{j^*}(h)$ and $s_{k^*}(h)$ such that $PN(h) = PN_{j^*k^*}(h)$. Thus Π' must have a point of tangency with $PN(h)$. But since $\Pi' \neq \Pi_3$, this means that Π' must leave $PN(h)$. Contradiction. □

INSERT FIGURE 9

Definition 3.20: A line (segment) L is a *gradient line (segment)* of V, if for every $x \in L$ such that $\nabla V(x) \neq 0$, one has that $\nabla V(x)$ is parallel to L.

Assumption 8: *There exist gradient line segments* L_i, $i = 1,2$ *of* V *which intersect the origin. Furthermore* L_1 *intersects* $u(h,0)$ *and* L_2 *intersects* $v(h,0)$ *for all* $h \in [h^-,h^+]$.

The following proposition is obvious.

Proposition 3.21: *If* $h \in [h^-,0]$ *then there exist bounded orbits* $\Pi_i(h)$ *such that*

(a) $P_1(\Pi_1(h)) \subset L_1$ *and intersects both* EO(h) *and* ET(h).

(b) $P_1(\Pi_2(h)) \subset L_2$ *and intersects both* EO(h) *and* EB(h).

Furthermore, if $h < 0$ *then* $\Pi_i(h)$ *is a periodic orbit and if* $h = 0$ *then* $\Pi_i(h)$ *is a homoclinic orbit with the origin as the critical point.*

Let $V: (0,90] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, $\Phi \in (0,90]$, and write $V_\Phi(x) = V(\Phi,x)$. Assume that for fixed Φ, V_Φ satisfies Assumptions 1, 2 and 4-8 where Φ equals the angle in degrees between L_1 and L_2. Again fixing Φ, define $\alpha_\Phi(s)$, $\beta_\Phi(s)$ and $\Psi_\Phi(s)$ as before for the potential function V_Φ.

Proposition 3.22: Given V_Φ as above, $\Psi_\Phi(s)$ is a continuous function on $(0,90] \times (0,1]$. In addition, for fixed Φ, Ψ_Φ is continuous on $[0,1]$.

Proof: The fact that Ψ_Φ is continuous on $[0,1]$ follows from the construction of Ψ_Φ and the definition of $\Psi_\Phi(0)$.

For fixed Φ, one has the partial parametrization of ET(0) and EB(0) corresponding to V_Φ. Denote these by u_Φ and v_Φ. Since V is continuous on $(0,90] \times \mathbb{R}^2$, one can choose u_Φ and v_Φ to be continuous on $(0,90] \times [0,1]$. Showing that Ψ is continuous on $(0,90] \times (0,1]$ is equivalent to showing that α and β are continuous on this region. Because α and β are similar, it is enough to show that α is continuous.

$$\alpha_\Phi(s) = \cos^{-1} \frac{<P_2(u_\Phi(s) \cdot \lambda(u_\Phi(s))),(1,0)>}{\|P_2(u_\Phi(s) \cdot \lambda(u_\Phi(s)))\|}.$$

Hence, it is sufficient to show that $u_\Phi(s) \cdot \lambda(u_\Phi(s))$ is continuous on $(0,90] \times (0,1]$. Because u is continuous, given $\epsilon > 0$, there exists $\delta > 0$, such that if $\|(\Phi,s) - (\Phi_0,s)\| < \delta$ then $\|u_\Phi(s) - u_{\Phi 0}(s)\| < \epsilon$. Now by the standard theorems on the continuity of initial conditions for solutions of ODE's one has that $u_\Phi(s) \cdot \lambda(u_\Phi(s))$ is continuous. □

Theorem 3.23: *Let* V *be as above. Let* $s^*(\Phi)$ *be the unique solution to* $\Psi(\Phi,s) = \pi$. *Then* $\lim_{\Phi \to 90^\circ} s^*(\Phi) = 0$.

Proof: What needs to be shown is that, given $\epsilon > 0$ there exists $\delta > 0$ such that if $|\Phi - 90| < \delta$ then $|s^*(\Phi)| < \epsilon$. Let $s_0' = \epsilon/2$, then $\Psi(90,s') = \pi - \delta$ where $\delta_0 > 0$. By Proposition 3.22, Ψ is continuous at $(90,s')$, hence there exists $\epsilon_0 < \epsilon/2$ such that, if $\|(\Phi,s) - (90,s')\| < \epsilon_0$ then $\Psi(\Phi,s) < \pi$. But Proposition 3.10 states that $\Psi(\Phi,0) > \pi$ hence $s^*(k) < s' + \epsilon_0 < \epsilon$ if $|\Phi - 90| < \delta$. □

It was shown earlier (Theorem 3.5) that if the angle Φ between the projection of the two homoclinic orbits $\Pi_i(0)$, $i = 1,2$, onto x-space is less than 90°, then there exists a unique periodic orbit $\Pi_3(0)$. Theorem 3.23 says that as this angle goes to 90°, the periodic orbit $\Pi_3(0)$ collapses onto the two homoclinic orbits, $\Pi_1(0)$ and $\Pi_2(0)$.

4. Crossing Orbits

Let V be a potential function as in the previous chapters satisfying Assumptions 1 - 8. Furthermore, for the sake of simplicity assume that the positive x_1-axis is a gradient line of V. Let $P_1(h)(\Pi_i(h)) = \pi_i(h)$ for $i = 1,2,3$. Let $J(h)$, $h \in [h^-,h^+]$ be the compact region in $\mathbb{R}^2$ with boundaries given by $EO(h)$, $ET(h)$, L_i for $i = 1,2$, and $\pi_3(h)$. (See Figure 10). Let $\mathcal{J}(h) = P_1^{-1}(h)J(h)$. Given $z \in \mathbb{R}^4$ define $\theta(z) = z \cdot \mathbb{R}$ and $\theta_i(z) = P_i(\theta(z))$, $i = 1,2$.

Definition 4.1: Let $H(z) = h$. Assume that there exists t_0 and t_1 such that $t_0 < t_1$, $P_1(z \cdot t_0) \in \pi_i(h)$, $P_1(z \cdot t_1) \in \pi_j(h)$, and $z \cdot [t_0,t_1] \in \mathcal{J}(h)$. Then one denotes z by $z_{ij}(h)$ and $z_{ij}(h)$ is called a *crossing orbit from* $\pi_i(h)$ *to* $\pi_j(h)$.

INSERT FIGURE 10.

In use $z_{ij}(h)$ will be taken to mean both an orbit with the above property and a generic point on such an orbit. Notice that $\theta_1(z_{ij}(h))$ intersects $\pi_i(h)$ and $\pi_j(h)$ transversally. In particular, the orbit originates outside of $J(h)$, enters $J(h)$ via $\pi_i(h)$, and exits $J(h)$ via $\pi_j(h)$.

Theorem 4.2: *There exists* $z_{ij}(0)$ *for* $i,j = 1,2,3$, *(except* $i = j = 3$*). Furthermore, there exists* $\tilde{v}(0) \in EB(0)$ *and* $\tilde{u}(0) \in ET(0)$ *so that* $\tilde{v}(0) = z_{11}(0)$ *and* $\tilde{u}(0) = z_{22}(0)$.

Proof: Let $\bar{u} = u(0,\bar{s})$. Then for $\bar{s}$ sufficiently close to 0, the orbit of $\bar{u}$ lies in an ϵ-tube about γ_+ over the interval $[0,t^*]$. (See Definitions 2.17 and 2.18.) Hence there exists $t_1 > 0$ such that $P_1(\bar{u}\cdot t_1) \in \pi_2(0)$. (See Figure 11(a).) Since the system is reversible, letting $t_0 = -t_1$, one has $P_1(\bar{u}\cdot t_0) \in \pi_2(0)$. Clearly, $P_1(\bar{u}\cdot[t_0,t_1]) \subset w(0)$. Therefore, $\bar{u} = z_{22}(0)$.

The proof that there exists $\bar{v}(0) = z_{11}(0)$ is similar.

To find z_{3j}, where $j = 1,2$, notice that because the positive x_1-axis is a gradient line there exists $z \in J(0)$, such that $P_1(z) = (x_1,0)$, where $x_1 > 0$ and z is on the stable manifold of the origin. By Definition 2.5 one has that $\lim_{t\to\infty} z\cdot t = (0,\xi)$ where $\xi = (\xi_1,0,-\xi_1,0)$ and hence one can define $\gamma_\pm(z)$ as usual. Now choose $\tilde{z}$ close to z, so that the orbit of $\tilde{z}$ lies in an ϵ-tube about $\gamma_+(z)$ on the interval $[0,t^*]$, but so that $\tilde{z}$ is not on the stable manifold. Then $\tilde{z} = z_{32}$. Similarly, choosing $\tilde{z}$ close to z, again, not on the stable manifold, but lying in an ϵ-tube about γ_-, gives z_{31}. (See Figure 11(b)). The reversibility of the system implies the existence of z_{13} and z_{23}.

Recall Assumption 8. Let the slope of $L_i = m_i$ where $i = 1,2$. By Assumption 3, $-1/m_1 < m_2$. So choose z on the stable manifold of the origin so that $P_1(z) = (x_1,mx_1)$ where $m > m_1$ and $-1/m < m_2$. (See Figure 11(c)). As before, construct the γ_+ curve corresponding to z. Then there exists $\tilde{z}$ close to z, so that the orbit of $\tilde{z}$ lies in a $[0,3]$ ϵ-tube about γ_+. Since $z \in J(0)$, $\tilde{z}$ can be chosen such that $\tilde{z} \in J(0)$. Now one readily checks that there exists $t_0,t_1 > 0$ such that $P_1(\tilde{z}\cdot t_0) \in \pi_1(0)$, $P_1(\tilde{z}\cdot t)$ $\in \pi_2(0)$ and $\tilde{z}\cdot[t_0,t_1] \subset J(0)$. Thus $\tilde{z} = z_{12}(0)$. Again, the reversibility of the system implies the existence of z_{21}. □

INSERT FIGURE 11.

Theorem 4.3: *Let* $h \in [h^-,h^+]$. *Then there exists* $z_{ij}(h)$ *where* $i,j = 1,2,3$ *(except* $i = j = 3$*). If* $h \in [h^-,0)$ *then* $z_{33}(h)$ *exists. Furthermore, one can choose* $\tilde{v}(h) \subset EB(h)$ *and* $\tilde{u}(h) \subset ET(h)$ *so that* $\tilde{v}(h) = z_{11}(h)$ *and* $\tilde{u}(h) = z_{22}(h)$.

Proof: If $h < 0$ then choose z so that $H(z) = h$, $P_1(z) = (x_1,0)$ and $x_1 > 0$ and $P_2(z) = (y_1,0)$ where $y_1 < 0$. Then $z = z_{33}(h)$.

Theorem 4.2 gives the existence of $z_{ij}(0)$. Having chosen a particular $z_{ij}(0)$, notice that $\theta(z_{ij}(0))$ is bounded away from the origin and that $\theta_1(z_{ij}(0))$ intersects $\pi_i(0)$ and $\pi_j(0)$ transversally. Thus for $|h^-|$ and $|h^+|$ sufficiently small, one can invoke the continuity of the flow to insure that there exists $z_{ij}(h)$.

The existence of $\tilde{v}(h) = z_{11}(h)$ and $\tilde{u}(h) = z_{22}(h)$ also follows from continuity. □

Theorem 4.4: *For* $\bar{x}_1 > 0$, *but sufficiently small it is possible to choose* $z_{ij}(0)$, *for* $i,j = 1,2,3$ *(except* $i = j = 3$*), such that* $(\bar{x}_1,0) \in \theta_1(z_{ij}(0))$.

Proof: Recall the proof of Theorem 4.2. In each case, z_{ij} was shown to exist by choosing a $\bar{z}$ whose orbit lay inside an appropriate $\gamma_\pm$ ϵ-tube. But each such ϵ-tube contains orbits which lie arbitrarily close to the origin and hence the positive x_1-axis intersected with each ϵ-tube gives an open interval of the form $(0,a)$. Choosing the minimum of these five a's gives an interval $(0,\tilde{a})$ such that if $\tilde{x}_1 \in (0,\tilde{a})$ then there exists z_{ij} such that $(\tilde{x}_1,0) \in \Theta_1(z_{ij})$. □

Corollary 4.5: *For* $h \in [h^-,h^+]$, *one can choose* $z_{ij}(h)$, *where* $i,j = 1,2,3$ (*except for* $i = j = 3$), *such that there exists* $\tilde{x}_1(h) > 0$ *with* $(\tilde{x}_1(h),0) \in \Theta_1(z_{ij}(h))$. *If* $h \in [h^-,0)$ *then the same is true for* $i = j = 3$.

From now on it is assumed that $z_{ij}(h)$ is chosen in this manner.

In what follows the techniques of Rod [7] are used extensively. It is assumed that the reader is familiar with [7] or has a copy readily available. A slight modification of his work is necessary since $\Pi_i(0)$, $i = 1,2$ are homoclinic orbits rather than periodic orbits. Thus we are forced to regenerate his definitions and lemmas in this different setting. We shall use the notation of [7] and refer to the proofs therein whenever possible.

The orbits $\theta(z_{ij}(h))$ will be used to divide $J(h)$ into three regions, which in turn can be used to classify the solutions of (2.12) which intersect $J(h)$.

Notation 4.6: For $h \in [h^-,h^+]$, let

$$\bar{D}_1(h) = \{x = (x_1,x_2) \mid x \in \Theta_1(\tilde{v}(h)) \text{ and } x_2 \leq 0\} \subset J(h).$$

$$\bar{D}_2(h) = \{x = (x_1,x_2) \mid x \in \Theta_1(\tilde{u}(h)) \text{ and } x_2 \leq 0\} \subset J(h).$$

$$\bar{D}_3(h) = \{x = (x_1,x_2) \mid x \in \Theta_1(z_{33}(h)) \text{ and } x_2 \leq \tilde{x}_1(h)\}.$$

Strictly speaking $\bar{D}_3(h)$ cannot be defined as above if $h \geq 0$. So in this case let

$$\bar{D}_3(h) = \{(x_1,0) \mid 0 \leq x_1 \leq \tilde{x}_1(h)\}.$$

$$\bar{\Sigma}_i(h) = \bar{D}_j(h) \cup \bar{D}_k(h) \quad \text{for} \quad i \neq j \neq k \neq i$$

$$\Sigma_i(h) = P_1^{-1}(h)(\bar{\Sigma}_i(h)) \qquad i = 1,2,3$$

$$D_i(h) = P_1^{-1}(h)(\bar{D}_i(h)).$$

Let $h \leq 0$, then

$$\bar{\Sigma}_3^* = \{x = (x_1,x_2) \mid \text{if} \quad x_2 \geq 0 \quad \text{then} \quad x \in \theta_1(u(h,1)) \quad \text{and}$$
$$\text{if} \quad x_2 \leq 0 \quad \text{then} \quad x \in \theta_1(v(h,1))\}.$$

Referring to Figure 12, let $\bar{\Sigma}_i^*(h)$, $i = 1,2$, be the curves shown. In particular, $\bar{\Sigma}_1^*(h)$ connects $ET(h)$ with $EO(h)$ and $\bar{\Sigma}_2^*(h)$ connects $EB(h)$ with $EO(h)$. Let

$$\Sigma_i^*(h) = P_1^{-1}(h)(\bar{\Sigma}_i^*(h)).$$

$\bar{R}_i(h)$ is the compact region in $J(h)$ bounded by $\bar{\Sigma}_i(h)$ and $\bar{\Sigma}_i^*(h)$ for $i = 1,2,3$. (See Figure 12)

$$R_i(h) = P_1^{-1}(h)(\bar{R}_i(h)).$$

INSERT FIGURE 12.

Theorem 3.19 guarantees that $R_3(h)$ is an isolating neighborhood for $\Pi_3(h)$, $h \in [h^-,h^+]$.

Assumption 9: *There exists* $\bar{\Sigma}_i^*(h)$ *for* $i = 1,2$, *so that if* $h < 0$ *then* $R_i(h)$ *is an isolating block for* $\Pi_i(h)$. *If* $h = 0$ *then* $R_i(0)$ *is as in Figure* 13, *in which case, it is assumed that the only bounded orbits in* $R_i(0)$ *are* $\Pi_i(0)$ *and the fixed point at the origin. Furthermore, any orbit which is tangent to* $\Sigma_i(0)$ *or* $\Sigma_i^*(0)$ *lies either on the stable manifold, or on the unstable manifold of the origin, or leaves* $R_i(0)$ *in forward and backward time without entering the interior of* $R_i(0)$.

INSERT FIGURE 13.

The importance of this assumption is that $R_1(h)$ and $R_2(h)$ are isolating blocks for $\Pi_1(h)$ and $\Pi_2(h)$. For conditions on V which induce the existence of such isolating neighborhoods the reader is referred to Churchill, Pecelli, and Rod [1].

Lemma 4.7: *(Rod* [7, *Lemma* 3.1]*). Let* $i = 1,2,3$, $h \leq 0$.

(a) $D_i(h)$ *is a closed topological two-disk.*

(b) Each Σ_i *and each* Σ_i^* *is a topological two sphere.*

Definition 4.8: For $i = 1,2,3$, $h \leq 0$ let

$$b_i^+(h) = \{z \in \Sigma_i(h) \mid \text{ there exists } \epsilon > 0 \text{ with } z\cdot[0,\epsilon] \subset \text{int } R_i(h)\}$$

$$b_i^-(h) = \{z \in \Sigma_i(h) \mid \text{ there exists } \epsilon > 0 \text{ with } z\cdot[-\epsilon,0] \subset \text{int } R_i(h)\}$$

$$T_i(h) = \Sigma_i(h)/\{b_i^+(h) \cup b_i^-(h)\}.$$

Lemma 4.9: *For* $h \leq 0$, $i = 1,2,3$,

(a) the $b_i^{\pm}$ *are disjoint open hemispheres in* S_i *transverse to the flow with* $b_i^- = \{(x,-y) \mid (x,y) \in b_i^+\}$.

(b) The ***tangency set,*** T_i, *is homeomorphic to a circle.*

(c) The orbits through points of T_i *"bounce off the region* R_i *to the outside." (Except when* $h = 0$ *and* $i = 1,2$*).*

Proof: For $h < 0$ or $i = 3$ see [7, Lemma 3.3]. So consider $i = 1$, $h = 0$. By Lemma 4.7, D_2 and D_3 can be represented as in Figure 14 where $P_1(\partial D_i) = (\tilde{x}_1,0)$ and concentric circles project to single points in R^2 under P_{21}. Notice the orientation chosen for the y values. The dark lines represent the tangencies to D_2 and D_3 and thus $T_1(0)$ is homeomorphic to S^1.

The proof for $T_2(0)$ is similar except one considers D_1 and D_3. □

INSERT FIGURE 14.

Definition 4.10: For $i = 1,2,3$, $h \leq 0$, and $z \in b_i^+(h)$ set

$$\sigma_i^+(h,z) = \inf\{t > 0 \mid z \cdot t \in b_i^-(h)\};$$

for $z \in b_i^-(h)$ set

$$\sigma_i^-(h,z) = \sup\{t < 0 \mid z \cdot t \in b_i^+(h)\}$$

provided that the inf and sup exist. Define $\varphi_i^{\pm}(h)$: $b_i^{\pm}(h) \longrightarrow b_i^{\mp}(h)$ by $\varphi_i^{\pm}(h)(z) = z \cdot \sigma_i^{\pm}(h,z)$ where the domain of $\varphi_i^{\pm}(h)$ is the same as that for $\sigma_i^{\pm}(h)$.

Lemma 4.11: *(Rod* [7, *Lemma* 3.4]*) For* $h \in [h^-,0]$, $i = 1,2,3$, $\sigma_j^{\pm}(h)$ *is continuous where defined with the domain being an open subset of* $b_i^{\pm}(h)$. $\varphi_i^{\pm}(h)$ *is a homeomorphism from domain to range with inverse* $\varphi_i^{\mp}(h)$.

Definition 4.12: For $h \in [h^-,0]$, $i = 1,2,3$, define

$$T_i^+(h) = \{z \in b_i^+ \mid z \cdot (0,\infty) \cap \Sigma_i^* \neq \varphi \quad \text{and} \quad z \cdot (0,\infty) \cap \Sigma_i = \varphi\}$$
$$T_i^-(h) = \{z \in b_i^- \mid z \cdot (-\infty,0) \cap \Sigma_i^* \neq \varphi \quad \text{and} \quad z \cdot (-\infty,0) \cap \Sigma_i = \varphi\}.$$

Notice that $T_i^{\pm}(h)$ consists of those points whose orbits pass from Σ_i through R_i and leave via Σ_i^* in $\pm$ time.

Definition 4.13: Let $NT_i^{\pm}(h)$, $i = 1,2,3$, $h \in [h_-,0]$, be the domain in $b_i^{\pm}(h)$ of the mapping $\varphi_i^{\pm}(h)$.

Definition 4.14: For $i = 1,2,3$, $h \in [h^-,0]$, let

$$a_i^+(h) = \{z \in b_i^+(h) \mid z \cdot [0,\infty) \subset R_i\}$$
$$a_i^-(h) = \{z \in b_i^-(h) \mid z \cdot (-\infty,0] \subset R_i\}.$$

In addition for $i = 1,2$, the origin is included in $a_i^{\pm}(0)$.

It should be clear that

Lemma 4.15: *For* $i = 1,2,3$, $h \in [h^-,0)$, $T_i^{\pm}(h)$, $NT_i^{\pm}(h)$ *and* $a_i^{\pm}(h)$ *are disjoint. This is also true for* $i = 3$ *and* $h = 0$. *If* $i = 1,2$, *and* $h = 0$, *then this holds except that* $a_i^+(0) \cap a_i^-(0)$ *is the origin.*

It should also be clear that any orbit in $J(h)$ which passes through $T_i^{\pm}(h)$ must leave $J(h)$ in forward or backward time. Should an orbit belong to $a_i^{\pm}(h)$, then that orbit must be bounded in positive or negative time. Finally, any orbit which belongs to $NT_i^{\pm}(h)$ cannot leave $J(h)$ through $R_i(h)$ immediately. Thus analyzing how orbits pass through $J(h)$ can be reduced to examining the orders in which the orbits can intersect $T_i^{\pm}(h)$, $NT_i^{\pm}(h)$, and $a_i^{\pm}(h)$.

Lemma 4.16: *(Rod, [7, Lemma 3.5 and following comments]). For* $i = 1,2,3$, *and* $h \in [h^-,0]$:

(a) The $T_i^{\pm}(h)$ *are homeomorphic open disks in* Σ_j.

(b) The $NT_i^{\pm}(h)$ *are homeomorphic open sets in* Σ_j.

(c) The $a_i^{\pm}(h)$ *are homeomorphic subsets in* Σ_j.

Lemma 4.17: *(Rod, [7, Lemma 3.6]). For* $i = 1,2,3$ *and* $h \in [h^-,0)$ or $i = 3$ *and* $h = 0$:

(a) The boundary of T_i^+, ∂T_i^+, *is a continuum which separates* Σ_i.

(b) NT_i^+ *is an open annulus with boundary* $T_i \cup \gamma_i$, *where* γ_i *is a continuum which separates* NT_i^+ *from* T_i^+ *in* b_i^+.

(c) $a_i^+(h) = \gamma_i(h) \cup \partial T_i^+(h)$ *is a continuum which separates* Σ_i *into two components.*

Assumption 10: *For* $h \in [h^-,0)$ *and* $i = 1,2,3$, $a_i^{\pm}$ *is the intersection of a sequence of closed annuli* $A_m^{\pm}(i) \subset b_i^{\pm}$, *each containing* $a_i^{\pm}$ *in its interior, with* $A_{m=1}^{\pm}(i) \subset \mathrm{int}(A_m^{\pm}(i))$ *for* $m = 1,2,3,\dots$.

Assumption 10 will be used in the following chapter to describe the behavior of orbits passing through $J(h)$ for $h < 0$. Again, for conditions on V which induce the existence of such annuli, the reader is referred to [1]. To see that these conditions actually imply Assumption 10, see [3].

5. Bounded Orbits

In this section the potential function V satisfies Assumptions 1-10 and the positive x_1-axis is taken to be a gradient line. The results are presented as a classification of the solutions to (2.12) which intersect $\mathcal{J}(h)$ for $h \in [h^-,h^+]$. This classification takes the form of a description of the order in which the orbits pass through $R_i(h)$ for $i = 1,2,3$. As will be seen the classification changes dramatically depending on whether $h > 0$, $h = 0$ or $h < 0$. The value of this example, however, comes from viewing h as a bifurcation parameter. In particular, $h = 0$ is a bifurcation point where the set of bounded orbits changes from a single periodic orbit to a "pathology" of bounded orbits.

5.1. $h \in (0,h^+]$

Since h is assumed fixed and greater than zero, it will be dropped from the notation.

Theorem 5.1: *For* $h \in (0,h^+]$ *the only bounded orbit in* $\mathcal{J}$ *is* Π_s.

Proof: This theorem is actually a corollary of Theorem 3.19 for the case $h > 0$. Recall that in the proof an isolating neighborhood PN_{12} is constructed. In this case we can choose $s_1 = 0$ and $s_2 = 1$. Hence $\mathcal{J} = PN_{12}$. □

5.2. h = 0

For this case we return to the ideas of Rod [7], however, it will become more evident where the differences lie between the description of the bounded orbits in his example and ours. The first step is to describe the behavior of the solutions to (2.12) as they cross the disks D_1, D_2 and D_3. Notice that $\partial D_1 = \partial D_2 = \partial D_3$.

Definition 5.2: For $i = 1,2$, let $e_i(1)$ and $e_i(2)$ be the first point and last point, respectively, at which $\theta(z_{ij})$ intersects D_3. Let $e_3(1) = (\bar{x}_1, 0, -\bar{y}_1, 0)$ where $\bar{x}_1$ is as in Theorem 4.4 and $\bar{y}_1 > 0$. Finally, let $e_3(2) = (\bar{x}_1, 0, \bar{y}_1, 0)$.

Using Corollary 4.5 and keeping in mind the conventions used in the proof of Lemma 4.9, one can represent D_i, $i = 1,2,3$ as is done in Figure 15.

Furthermore, $\theta(z_{11})$ divides D_1 into a closed right half disk, RD_1, and a closed left half disk LD_1. Similarly $\theta(z_{22})$ divides D_2 into RD_2 and LD_2. Finally, D_3 is divided into a closed upper half disk, UD_3, and a closed lower half disk, LD_3, by the orbits on the stable and unstable manifold of the origin which project onto the positive x_1-axis. In addition, notice that except for $\theta(z_{11})$, RD_1 consists of the orbits which leave R_2 and immediately enter R_3, and LD_1 consists of the orbits which leave R_3 and immediately enter R_2. Similar statements can be made for LD_2, RD_2, LD_3 and UD_3.

INSERT FIGURE 15.

We are interested in describing how $a_i^{\pm}$, $T_i^{\pm}$ and $NT_i^{\pm}$ intersect D_j for $i,j = 1,2,3$. For our purposes it is sufficient to show that the geometric information present in Figure 16 is correct. To do so we shall restrict our attention to RD_1 and UD_3 and claim that the arguments for the other half disks are similar. Our strategy is to determine the sets $\partial RD_1 \cap T_i^{\pm}$ and $\partial UD_3 \cap T_i^{\pm}$ and then use results from Section 4 to obtain Figure 16. To do this the following notation will be useful. If $a,b \in \partial RD_1$ (∂UD_3) then (a,b) denotes the open segment of ∂RD_1 (∂UD_3) obtained by starting at a and proceeding to b along ∂RD_1 (∂UD_3) in a clockwise direction. $[a,b]$ denotes the corresponding closed segment.

INSERT FIGURE 16.

We begin with two technical lemmas which, are relavent since $D^2V(0) = I$.

Lemma 5.3: *Let* $z = (x_0,0,0,x_0)$ *where* $x_0 > 0$. *Let* $\tilde{z} = (x_0,0,y_1,y_2)$ *where* $y_1 < 0$, $y_2 > 0$ *and* $H_L(\tilde{z}) = 0$ (H_L *is defined as in* (2.2)). *Consider* $\theta(z)$ *and* $\theta(\tilde{z})$ *solutions to the linear equations* (2.3). *Then* $\theta_1(z) \cap \theta_1(\tilde{z}) = (x_0,0)$.

Proof: The exact solution to (2.3) is given by $z(t) = \exp(tA)z(0)$. More explicitly, we have

$$\exp(tA) = \begin{bmatrix} \Theta(t) & o & \Psi(t) & o \\ o & \Theta(t) & o & \Psi(t) \\ \Psi(t) & o & \Theta(t) & o \\ o & \Psi(t) & o & \Theta(t) \end{bmatrix}$$

where

$$\Theta(t) = \cos(it) = \frac{1}{2}(e^{t} + e^{-t})$$

and

$$\Psi(t) = -i \sin(it) = \frac{1}{2}(e^{t} - e^{-t}).$$

Thus a solution to $\theta_1(z) \cap \theta_1(\tilde{z})$ must be a solution to

$$\Theta(t)x_0 = \Theta(t_0)x_0 + \Psi(t_0)y_1 \tag{5.1}$$

and

$$\Psi(t)x_0 = \Psi(t_0)y_2. \tag{5.2}$$

Since $H_2(\tilde{z}) = 0$ and $y_1 < 0$ one has that

$$y_1 = -\sqrt{x_0^2 + y_2^2}\ . \tag{5.3}$$

Substituting (5.2) and (5.3) into (5.1) gives

$$(\Theta(t) - \Theta(t_0))x_0 = \Psi(t_0)x_0\ \sqrt{(1+\Psi^2(t))/\Psi^2(t_0)}$$

or

$$\Theta^2(t) - \Theta^2(t_0) = \psi^2(t_0) + \psi^2(t).$$

Expanding gives

$$2 - (e^{t+t_0} + e^{t-t_0} + e^{-t+t_0} + e^{-t-t_0}) = 0.$$

The only solution to this is $t = t_0 = 0$. □

Lemma 5.4: *Let* $z = (x_0,0,-x_0,0)$, $x_0 > 0$ *and* $\tilde{z} = (x_0,0,y_1,y_2)$ *where* $y_1 < 0$, $y_2 > 0$ *and* $H_L(\tilde{z}) = 0$. *Consider* $\theta(z)$ *and* $\theta(\tilde{z})$ *solutions to the linear equation* (2.3). *There* $\theta_1(z) \cap \theta_1(\tilde{z}) = (x_0,0)$.

Proof: The proof is similar to that of Proposition 5.3. □

Proposition 5.5: *If* $\|(\tilde{x}_1,0)\|$ *is small enough then* $z \in (e_3(1), z_{21}] \subset T_1^+$.

Proof: Choosing $\|(\tilde{x}_1,0)\|$ small means that one can approximate the orbits $\theta_1(z)$ of the nonlinear flow (2.18) by the orbit of $\theta_1(z)$ of the linear flow (2.2). By Lemmas 5.3 and 5.4, $\theta_1(z)$ (linear) is bounded by the positive x_1-axis and $\theta_1(z_{21})$, both of which intersect $\Sigma_1^*(0)$. Thus $\theta_1(z)$ (nonlinear) crosses $\Sigma_1^*(0)$, i.e., $z \in T_1^+(0)$. □

Since the x_1-axis is a gradient line of V, $e_3(2) \in T_3^+$. Refering to the proof of Theorem 4.2 one can conclude that $[z_{23},z_{13}] \subset T_3^+$. By definition $e_2(1)$ leaves J via Σ_2^* thus $e_2(1) \notin T_3^+ \cup T_1^+$. Similarly $e_1(2) \notin T_3^+ \cup T_2^+$. Therefore, $\partial T_1^+ \cap (z_{21},e_2(1)) \neq \phi$, $\partial T_3^+ \cap (e_2(1),z_{23}) \neq \phi$ and $\partial T_3^+ \cap (z_{13},e_1(2)) \neq \phi$.

Using the reversibility of Hamiltonian systems we can also conclude that $(e_3(1),z_{31}] \subset T_3^-$, $[z_{21},e_3(2)) \subset T_2^-$, $(e_3(2),e_1(1)] \subset T_1^-$, $\partial T_2^- \cap (e_1(2),z_{21}) \neq \phi$

and $\partial T_3^- \cap (z_{31}, e_1(2)) \neq \phi$. Finally, by definition, for $i = 1,2$, $e_3(1) \in a_i^+$ and $e_3(2) \in a_i^-$.

With these general results established we can now turn to the specific examples. We begin with RD_1. Clearly $[e_1(1), e_2(1)] \subset T_1^+$ since it is defined by $\theta(z_{11})$. Combining this with proposition 5.5 gives that $[e_1(1), z_{21}] \subset T_1^+$. Lemma 4.16 says that $T_1^+ \cap RD_1$ is an open set in RD_1, hence there must be a component of this open set which contains $[e_1(1), z_{21}]$. Similarly, there is a component of $T_3^+ \cap RD_1$ which contains $[z_{23}, z_{13}]$. The boundaries of these two components are subsets of a_1^+ and a_3^+, respectively. But $a_1^+ \cap a_3^+ = \phi$. Thus we can represent $T_i^+ \cap RD_1$ and $a_i^+ \cap RD_1$, $i = 1,3$, as is done in Figure 16.

Determining how T_i^- intersects RD_1 is slightly more difficult since a_1^- and a_2^- both represent how the unstable manifold at the origin intersects RD_1 and hence, we do not have that $a_1^- \cap a_i^- = \phi$. Never the less, from the previous general results we can conclude that $e_3(2) \in \partial T_1^- \cap \partial T_2^-$. Since $[e_1(1), e_1(2)] \subset T_1^-$ and T_1^- is open in RD_1, we have that a_i^-, $i = 1,2$, is bounded away from $[e_1(1), e_1(2)]$. Away from $\theta(z_{11})$, RD_1 is transverse to the flow and thus $\partial T_1^- \cap \partial T_2^-$ (which represents a portion of the stable manifold to the origin) cannot branch apart. Thus $a_1^- = a_2^-$ and separates T_1^- and T_2^- in RD_1. Thus, again we have the result of Figure 16.

Now consider UD_3. Let e denote a line segment connecting O and z_{21} in UD_3. Recalling the proof of the existence of z_{21} (Theorem 4.2) one recognizes that if $z \in e \backslash \{O\}$ then $z \in T_1^+ \cap T_2^-$. $[O, z_{21}] \cup e$ bounds a region in UD_3 and Proposition 5.5 can be used to show that any element interior to this region is an element of T_1^+. $[e_3(2),O)$ is defined by the x_1-axis and hence is contained in T_3^+ thus by previous remarks $[e_2(1),O) \cap T_1^+ = \phi$. Therefore, we can conclude that for $i = 1,3$, T_i^+ and a_i^+ are as in Figure 16. A similar argument can be used for T_i^- and a_i^-, $i = 2,3$.

We are now in a position to consider the classification scheme and prove the existence, or lack thereof, of certain orbits. Our classification will be done by describing the sequence in which the orbits pass through the interiors of the R_i's, $i = 1,2,3$. Let s be a sequence $\{s_k\}$ (possibly bi-infinite) where $s_k \neq s_{k+1}$ and $s_k \in \{-\infty,1,2,3,\infty\}$. As will be shown to classify the orbits of (2.18) which intersect $J(h)$, $h \in [h^-,0)$, one needs the following 9 types of sequences:

(T1) s is a bi-infinite sequence $\{s_k\}_{k=-\infty}^{\infty}$ and $s_k \in \{1,2,3\}$.

(T2) $s = \{s_k\}_{k=0}^{\infty}$ and $s_k \in \{1,2,3\}$.

(T3) $s = \{s_k\}_{k=-\infty}^{0}$ and $s_k \in \{1,2,3\}$.

(T4) $s = \{s_k\}_{k=0}^{\infty}$ and $s_0 = -\infty$, $s_k \in \{1,2,3\}$ for $k \geq 1$.

(T5) $s = \{s_k\}_{k=-\infty}^{0}$ and $s_0 = -\infty$, $s_k \in \{1,2,3\}$ for $k \leq -1$.

(T6) $s = \{s_k\}_{k=0}^{n}$ and $s_k \in \{1,2,3\}$.

(T7) $s = \{s_k\}_{k=0}^{n}$ and $s_0 = -\infty$, $s_k \in \{1,2,3\}$ for $1 \leq k \leq n$.

(T8) $s = \{s_k\}_{k=0}^{n}$ and $s_n = \infty$, $s_k \in \{1,2,3\}$ for $1 \leq k < n$.

(T9) $s = \{s_k\}_{k=0}^{n}$ and $s_0 = -\infty$, $s_n = \infty$, and $s_k \in \{1,2,3\}$ for $0 < k < n$.

To see how these sequences describe orbits, notice that given $s = \{s_k\}$ one can associate a sequence $R(s) = \{R(s_k)\}$, where $R(s_k) = R_{s_k}$ if

$s_k \in \{1,2,3\}$. One says that an orbit follows s, if as time increases the orbit passes into the interior of each R_{s_k} successively. If $s_0 = -\infty$ as in (T4), (T5) and (T9), then one says that the orbit entered $R(s_1)$ via $\Sigma^*_{s_1}$. Similarly, if $s_n = \infty$ as in (T5), (T8) and (T9), then one says that the orbit left $R(s_{n-1})$ via $\Sigma^*_{s_{n-1}}$.

Though all sequences of the form (T1) - (T9) are necessary in the case $h < 0$, for $h = 0$ the results are much simpler. This is what one expects since $h = 0$ is the bifurcation point. As Theorem 5.6 demonstrates, the only sequences which appear fall into the types (T6) - (T9) and furthermore, most sequences s in (T6) - (T9) are not realized. However, comparing the results of Theorem 5.6 with those of Theorem 5.8 allows one to see how the appearance of the critical point at the origin "separates" orbits entering and exiting J via $\Sigma_1 \cup \Sigma_2$. This separation is the bifurcation which gives rise to the pathology of orbits in J.

Theorem 5.6: s *is a sequence representing an orbit on the energy level* $H = 0$ *if and only if* $s = \{s_k\}$ *satisfies*:

(1) *There exists at most one* k *such that* $s_k = 3$.

(2) *If* $\{s_1,s_2,s_3\} = \{1,2,1\}$ *or* $\{2,1,2\}$ *then* $s_0 = -\infty$ *and* $s_4 = \infty$.

(3) *If* $\{s_1,s_2,s_3\} = \{1,2,3\}$ *or* $\{2,1,3\}$ *then* $s_0 = -\infty$.

(4) *If* $\{s_1,s_2,s_3\} = \{3,2,1\}$ *or* $\{3,1,2\}$ *then* $s_4 = \infty$.

(5) $\{1,2\}$ *and* $\{2,1\}$ *are not possible sequences.*

Proof: We first show that if s represents an orbit then s satisfies (1) through (5). Let $\theta(z)$ be an orbit represented by s.

(1) We need to show that if $s_\ell = 3$ and $k \neq \ell$ then $s_k \neq 3$. Without loss of generality we can assume $s_0 = 3$ and we need only show that $s_k \neq 3$ for all $k > 0$. If $s_1 = \infty$ or s_1 does not exist then we are done. Thus $s_1 = 1$ or $s_1 = 2$. In either case the argument that follows is similar so let $s_1 = 2$. This implies there exists t_0 such that $z \cdot t_0 \in LD_1$. By Figure 16 one can conclude that $z \cdot t_0 \in T_1^+$, $z \cdot t_0 \in T_2^+$ or $z \cdot t_0 \in a_1^+ = a_2^+$. In the first case, by the definition of T_1^+, $s_2 = 1$ and $s_3 = \infty$. In the second case $s_2 = \infty$ and finally if $z \cdot t_0 \in a_1^+ = a_2^+$ then s_2 does not exist.

(2) The assumption that $\{1,2,1\} = \{s_1,s_2,s_3\}$ implies that there exists $t_0 < t_1$ such that $z \cdot t_0 \in LD_3$, $z \cdot t_1 \in UD_3$ and $z \cdot (t_0,t_1) \subset R_2$. Refering to Figure 16 we have that $z \cdot t_1 \notin T_2^-$ implies $z \cdot t_1 \in T_1^+$ and $z \cdot t_0 \notin T_2^+$ implies $z \cdot t_0 \in T_1^-$ thus $s = \{-\infty,1,2,1,\infty\}$.

(3) Assume $\{s_1,s_2,s_3\} = \{1,2,3\}$, then there exists $t_0 < t_1$ such that $z \cdot t_0 \in LD_3$, $z \cdot t_1 \in RD_1$ and $z \cdot (t_0,t_1) \subset R_2$. Since $s_3 = 3$, $z \cdot t_0 \in T_2^+$ hence $z \cdot t_0 \in T_1^-$ and therefore $s_0 = -\infty$. If $\{s_1,s_2 s_3\} = \{2,1,3\}$ the argument is similar.

(4) The argument is similar to that of (3).

(5) $s = \{1,2\}$ means that a_1^- intersects a_2^+ non trivially in $LD_3 \backslash \{O\}$. Figure 16 says this cannot happen.

Showing that, if s satisfies 1-5 then there exists an orbit whose path is represented by s, is easy but tedious. (1) - (4) implies that the length of s is less than or equal to 7. Thus there exists a finite number of possible orbit types. Checking that each orbit in fact exists is therefore possible and left to the reader. As an example we shall demonstrate the existence of $\Theta(z)$ given a particular s.

Let $s = \{-\infty,1,2,3,1,2,\infty\} = \{s_k\}_{k=0}^{6}$. We need to find z such that $\theta(z)$ is represented by s. Let K_0 be a closed set with non-empty interior such that $K_0 \subset LD_3 \setminus (T_3^+ \cup T_2^+ \cup a_2^+)$ and $K_0 \cap a_3^+ \neq \phi$. We can choose K_0 small enough so that $\varphi_2^+(K_0) \subset RD_1$. Thus if $z' \in K_0$ and s' represents $\theta(z')$ then $s' = \{-\infty,1,2,3,\ldots\}$. There exist subsets of K_0 which in forward time pass near $\Pi_3(0)$ and exit from R_3 into R_2. (See Rod [7] for details). Thus $\phi \neq K_1 \equiv \varphi_3^+ \circ \varphi_2^+(K_0) \cap D_2$. Furthermore, $K_1 \cap T_2^+ \neq \phi$. Thus if $z \in K_1$ and s represents $\theta(z)$ then $s = \{-\infty,1,2,3,1,2,\infty\}$. □

Corollary 5.7: *(a) If* $s = \{3,1\}$, $\{1,3\}$, $\{2,3\}$ *or* $\{3,2\}$ *then there exists at least one orbit* $\theta(z)$ *which is represented by* s.

(b) There exists at least countably many orbits of the type $s = \{1,3,1\}$, $\{2,3,2\}$, $\{1,3,2\}$ *or* $\{2,3,1\}$.

(c) There exists at least uncountably many orbits of the type s *if* s *is not of type (a) or (b).*

Proof: See Rod [7] for details. □

5.3. $h \in [h^-,0)$.

In this region we have three periodic orbits whose stable and unstable manifolds intersect transversely. For this case the work has been done for us by Churchill and Rod, [2] and [3]. Theorem 5.8 implies that the set of bounded solutions is much more complicated when

$h < 0$ than when $h = 0$, since any sequence of type (T1) - (T9) corresponds to an orbit passing through J.

Theorem 5.8: *(a) Let* s *be of type* (T1). *Then there exists uncountably many solutions of* (2.18) *which pass through the sequence of regions* R(s).

(b) Let s *be of type* (T2) - (T5) *or* (T7) - (T9). *Then there exist uncountably many solutions passing through the sequence* R(s).

(c) Let s *be of type* (T6) *with* $n \geq 3$. *Then there exist at least countably many solutions passing through* R(s).

Proof: This theorem follows from a collection of theorems in [2] and [3]. More specifically, (a) follows from Theorem 1.3 in [3] and (b) and (c) follow from Theorem 6.4 in [2]. □

Theorem 5.9: *Assume* s *is of type* (T1) *and periodic with period* n, *i.e.,* $s_k = s_{k+n}$ *for all* k *where* $n > 0$. *Then given any* $m > 0$, *there exists at least one periodic orbit which passes through the sequence*

$$\{s_k\}_{k=k_0}^{k_0+n}$$

m-*times and then closes up.*

Proof: See [3], Theorem 1.3. □

Both Theorem 5.8 and Theorem 5.9 are dependent on Theorem 1.3 of Churchill and Rod [3]. In their paper they give 6 hypotheses that must be satisfied in order for the theorem to hold. It is straightforward to check that the first two are satisfied. The third hypothesis follows by arguments similar to those of [3], Section 3. The fourth hypothesis is Assumption 10 in Chapter 4 of this paper. The fifth hypothesis is also satisfied by the results of Chapter 4. Hence only the sixth hypothesis needs to be demonstrated. This is the content of the following proposition.

Proposition 5.12: *Let* $U \subset T_i^-$ *be the maximal connected open (relative to* Σ_i*) set containing* $z_{ij} \in T$ *which is carried homeomorphically by the flow onto* $U^* \subset T_j^+$. *Also, let* $K \subset a_i^-$ *be any connected set intersecting the closure of* U *which is carried homeomorphically by the flow onto* $K^* \subset T_j^+ \subset a_j^+$. *Then:*

(a) The closure of U *is carried homeomorphically by the flow onto the closure of* U^*.

(b) The closure of K *is carried homeomorphically by the flow onto the closure of* K^*.

Proof: Let $\varphi: U \to U^*$ denote the homeomorphism. If $p \in U$ then $\varphi(p) = q \in U^*$ and there exist $t(p) > 0$ so that $p \cdot t(p) = q$. Let $\tilde{q}$ be in the closure of U^* and let $q_n \to \tilde{q}$ where $q_n \in U^*$ for all n. Then there exists $p_n \in U$ and $t(p_n) > 0$ such that $p_n \cdot t(p_n) = q_n$. The closure of U is compact, hence there exists a convergent subsequence $\{p_m\}$ such that $p_m \to \tilde{p}$ an element of the closure of U. If it can be shown that there exists a $t(\tilde{p})$ such that $\tilde{p} \cdot y(\tilde{p}) = \tilde{q}$, then (a) will be shown to be true. It is sufficient to show that $\{t(p_n)\}$ is bounded.

So consider the case $i = 1$, $j = 2$. Define $M(z) = (m_1,m_2,m_3)$ by letting m_i equal the number of times $\theta_1(z)$ crosses D_i for $i = 1,2$ and m_3 equal the number of times $\theta_1(z)$ crosses the positive x_1-axis. Since for $z \in U$, $\theta_1(z)$ always crosses each arc transversally, $M: U \to Z \times Z \times Z$ is continuous. But $M(z_{12}) = (0,0,1)$ thus $M(z) = (0,0,1)$ for all $z \in U$. If $\{t(p_n)\}$ is unbounded then either:

(a) there exists $p_n \in U$ such that $M(p_n) = (m_1,m_2,m_3)$ where either $m_3 > 1$ or $m_i > 0$ for $i = 1,2$, or

(b) $\theta(p_n)$ remains arbitrarily long in R_3.

Case (a) cannot happen since M is continuous and case (b) cannot happen since this would force m_3 to be arbitrarily large.

The proof for the other z_{ij}'s follows in a similar manner. The proof of (b) is also similar. □

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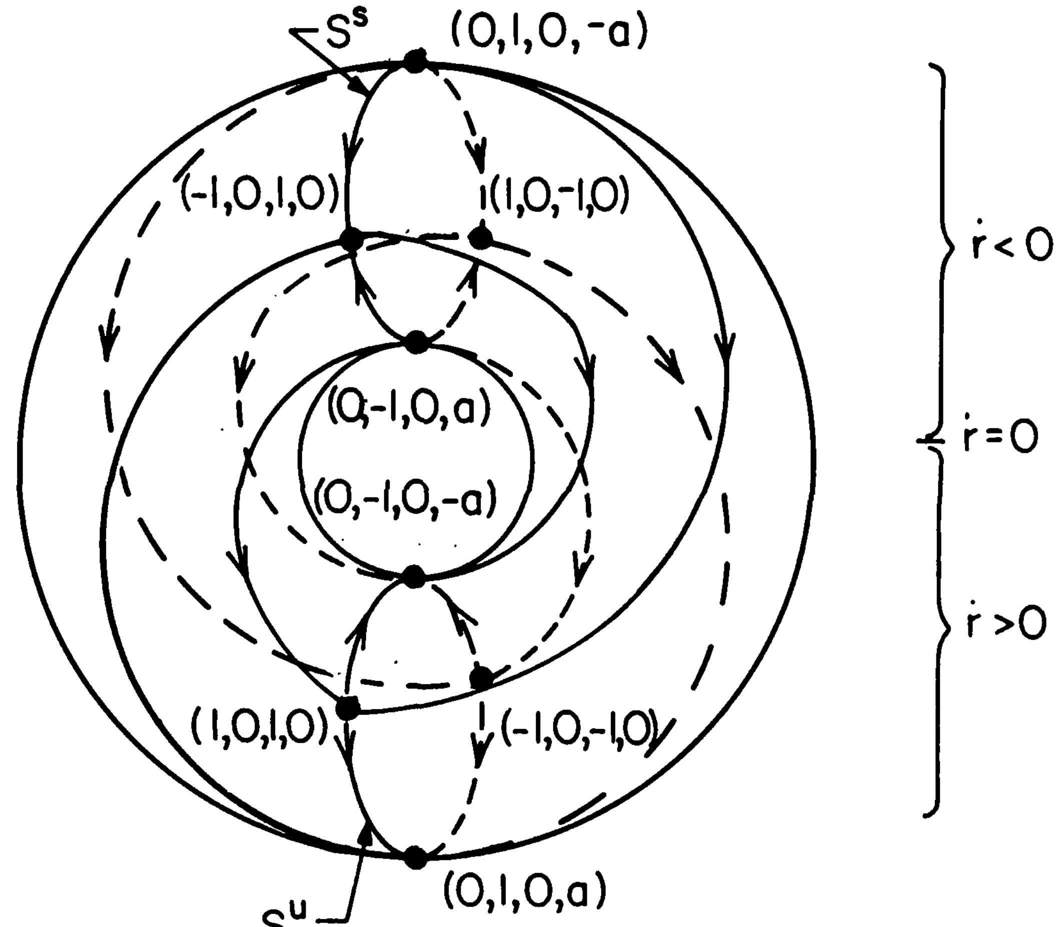
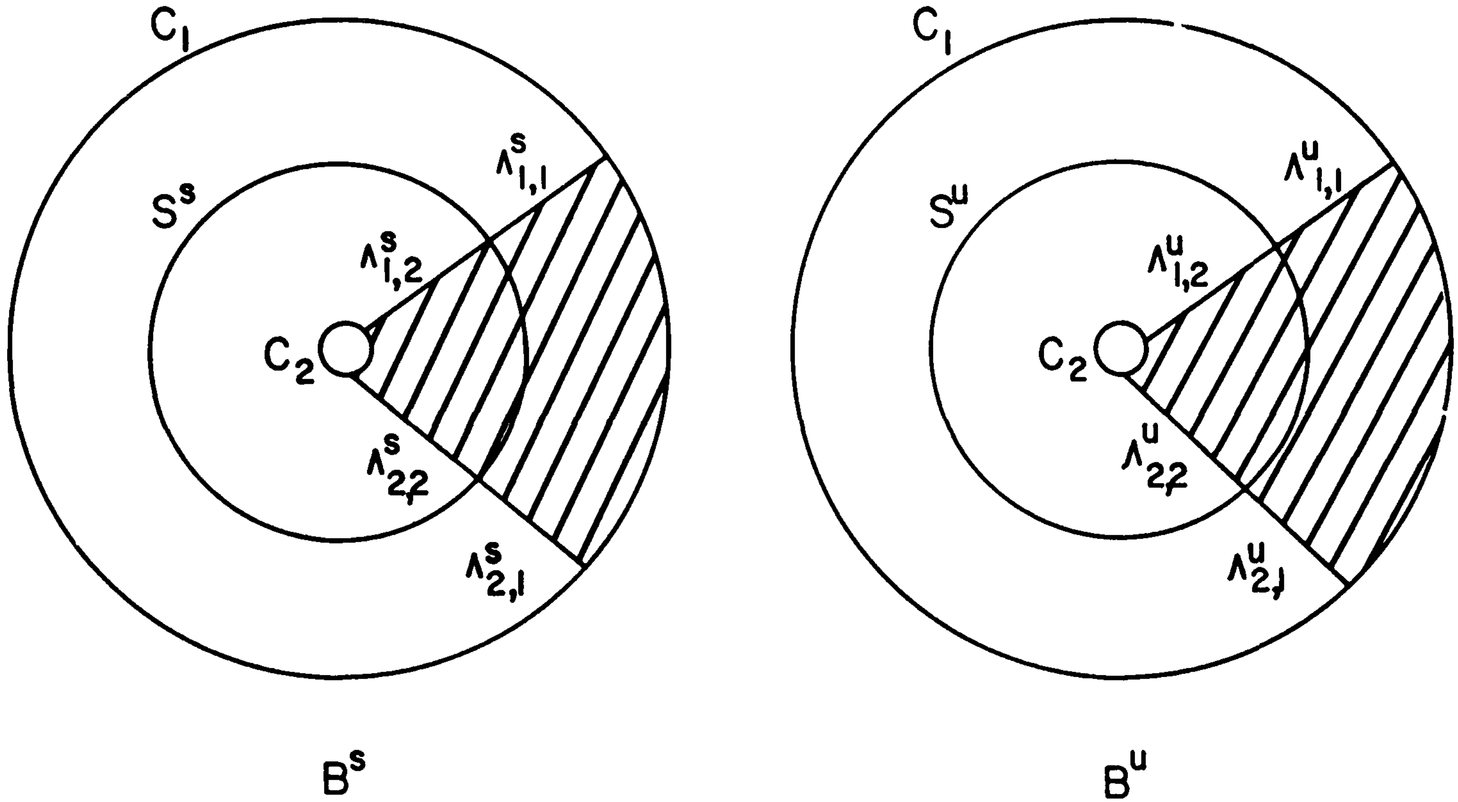
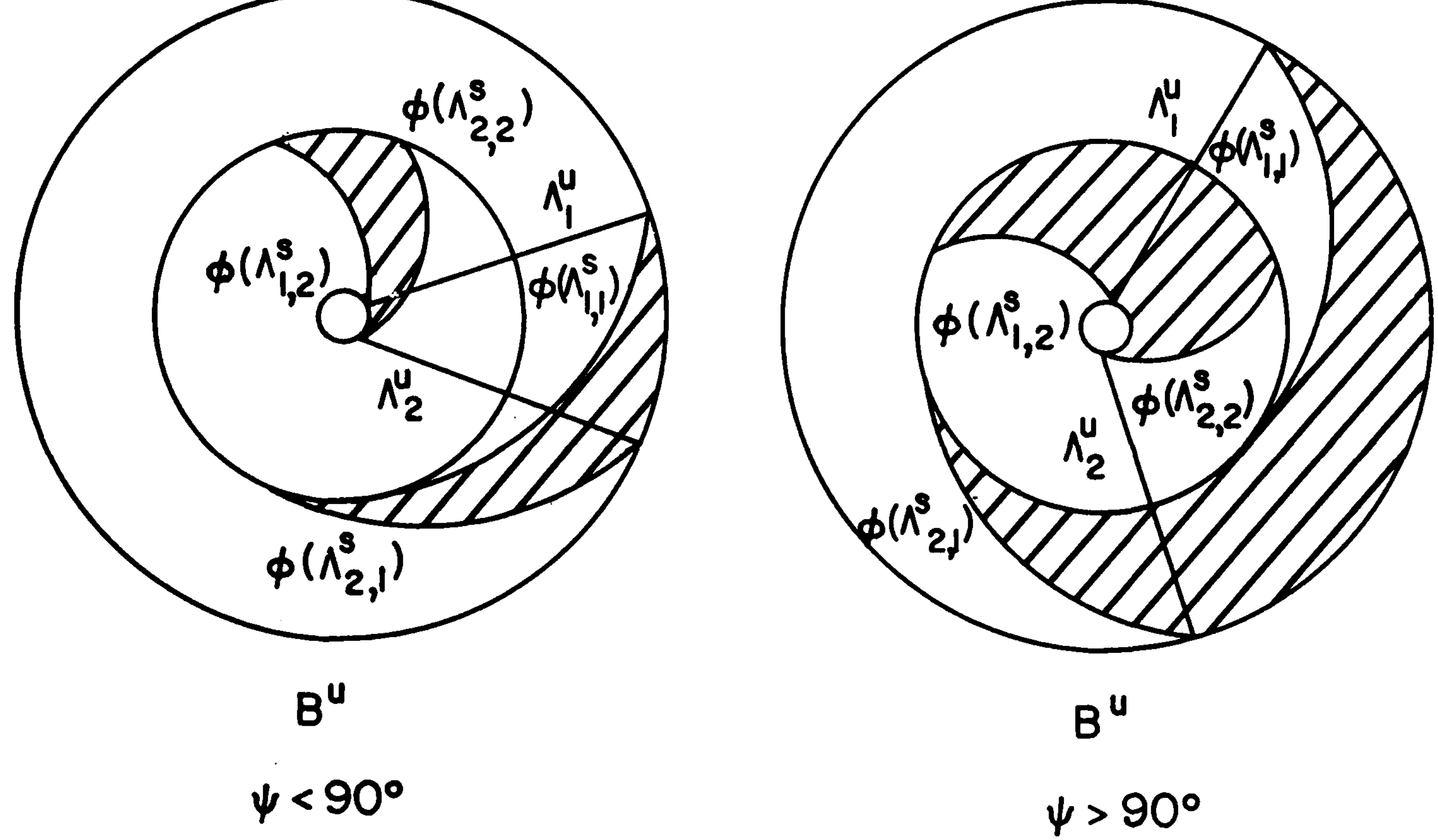

S^s
(0,1,0,-a)
(-1,0,1,0)
(1,0,-1,0)
(0,-1,0,a)
(0,-1,0,-a)
(1,0,1,0)
(-1,0,-1,0)
(0,1,0,a)
S^u
$\dot{r} < 0$
$\dot{r} = 0$
$\dot{r} > 0$

Figure 1



C_1
S^s
$\Lambda^s_{1,1}$
$\Lambda^s_{1,2}$
C_2
$\Lambda^s_{2,2}$
$\Lambda^s_{2,1}$
B^s
C_1
S^u
$\Lambda^u_{1,1}$
$\Lambda^u_{1,2}$
C_2
$\Lambda^u_{2,2}$
$\Lambda^u_{2,1}$
B^u

Figure 2



$\phi(\Lambda^s_{2,2})$
Λ^u_1
$\phi(\Lambda^s_{1,2})$
$\phi(\Lambda^s_{1,1})$
Λ^u_2
$\phi(\Lambda^s_{2,1})$
B^u
$\psi < 90°$
Λ^u_1
$\phi(\Lambda^s_{1,1})$
$\phi(\Lambda^s_{1,2})$
$\phi(\Lambda^s_{2,2})$
Λ^u_2
$\phi(\Lambda^s_{2,1})$
B^u
$\psi > 90°$

Figure 3

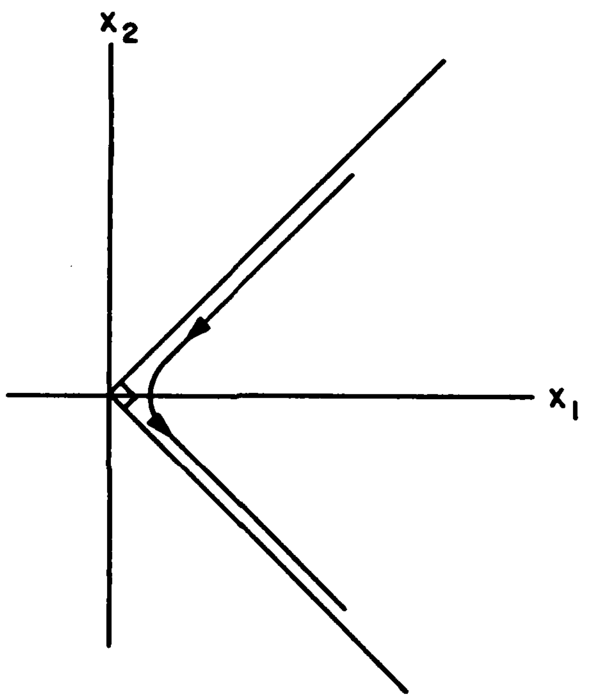


Figure 4

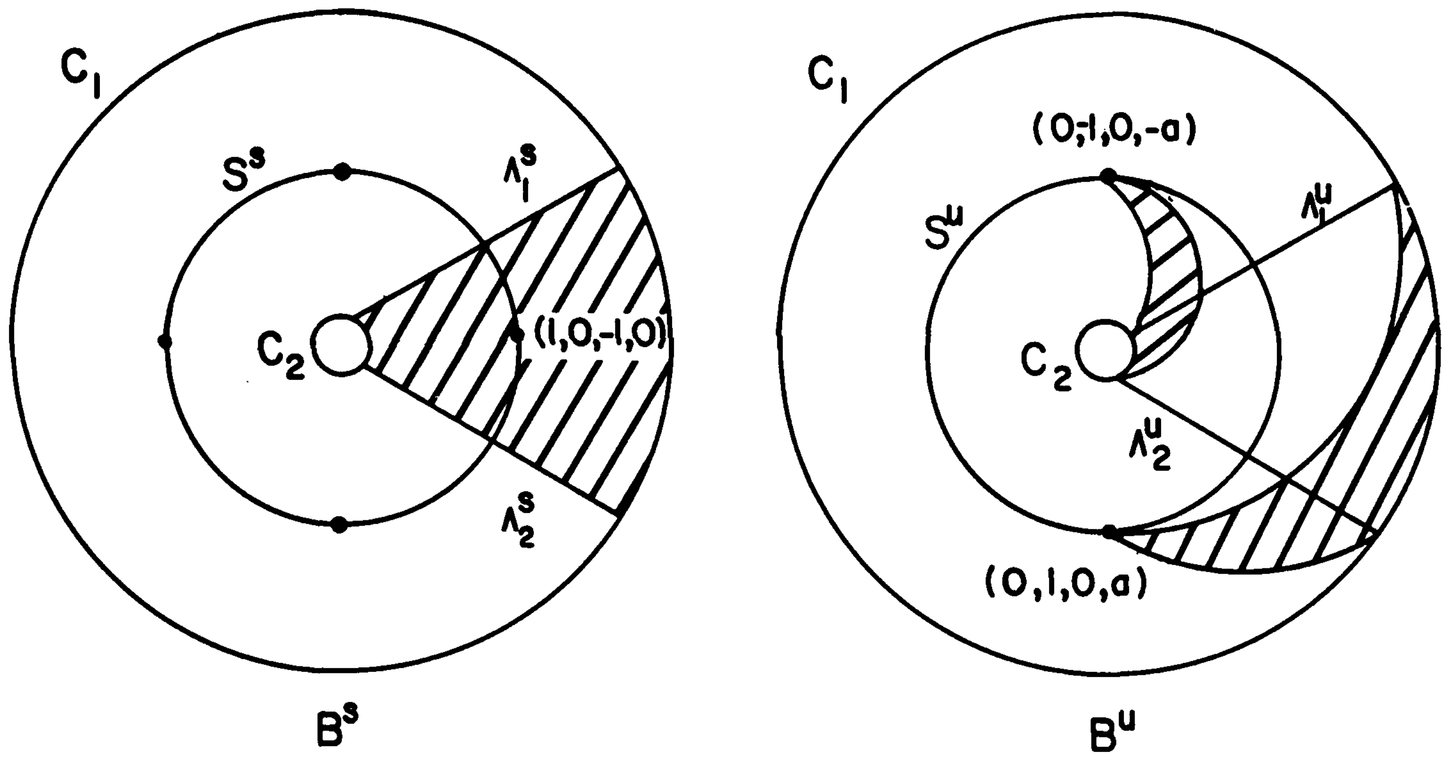


Figure 5

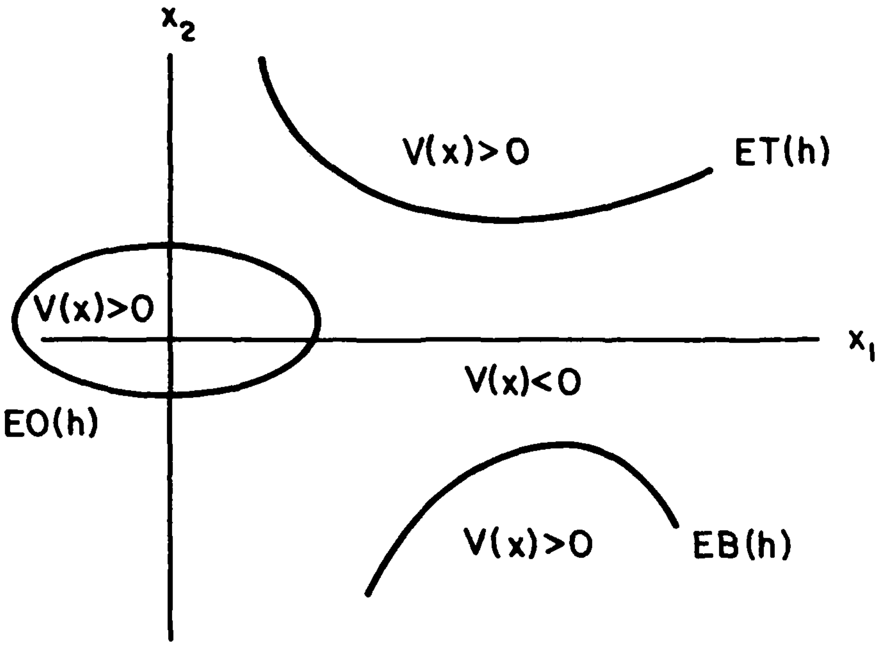


Figure 6

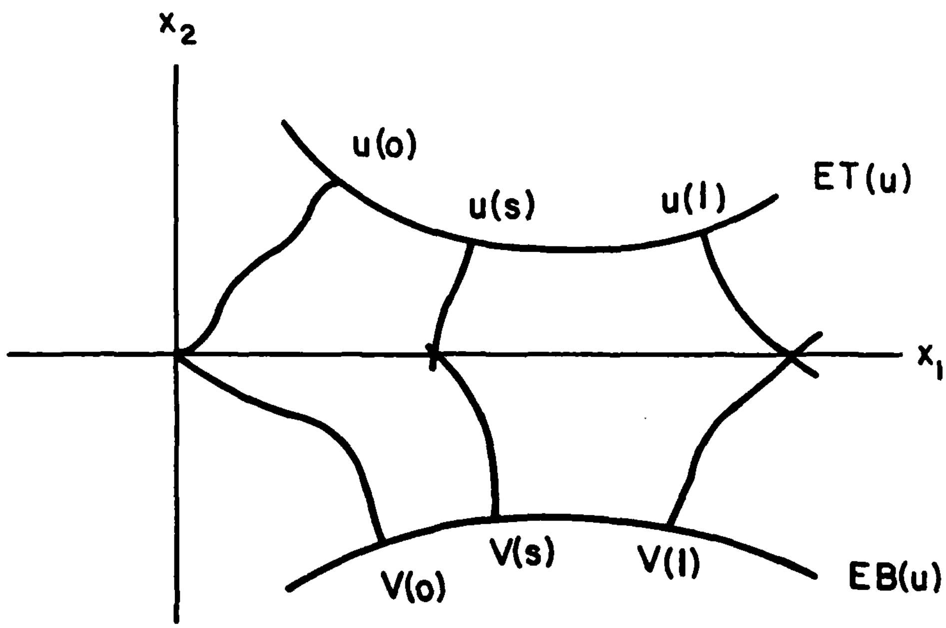


Figure 7

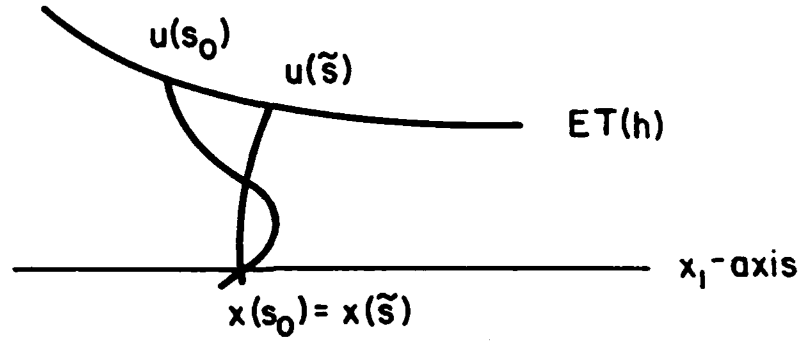


Figure 8

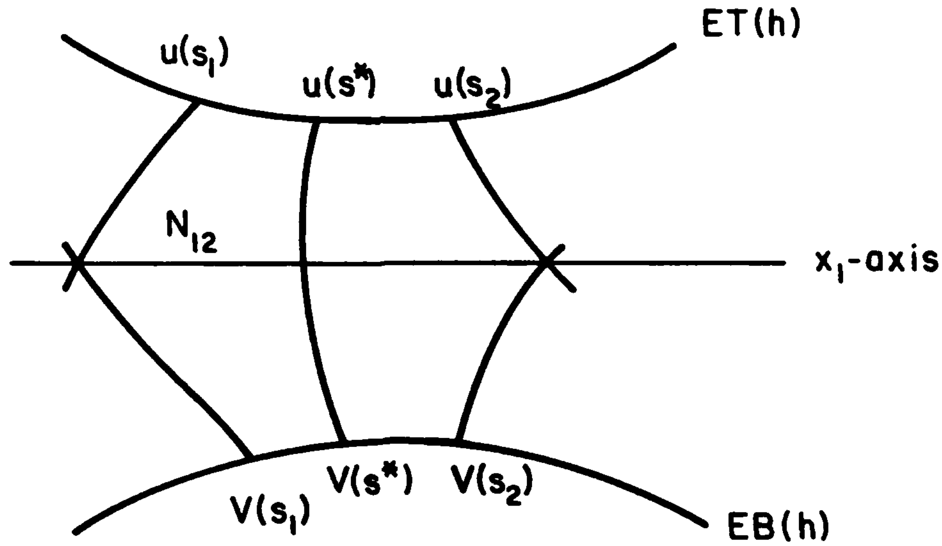


Figure 9

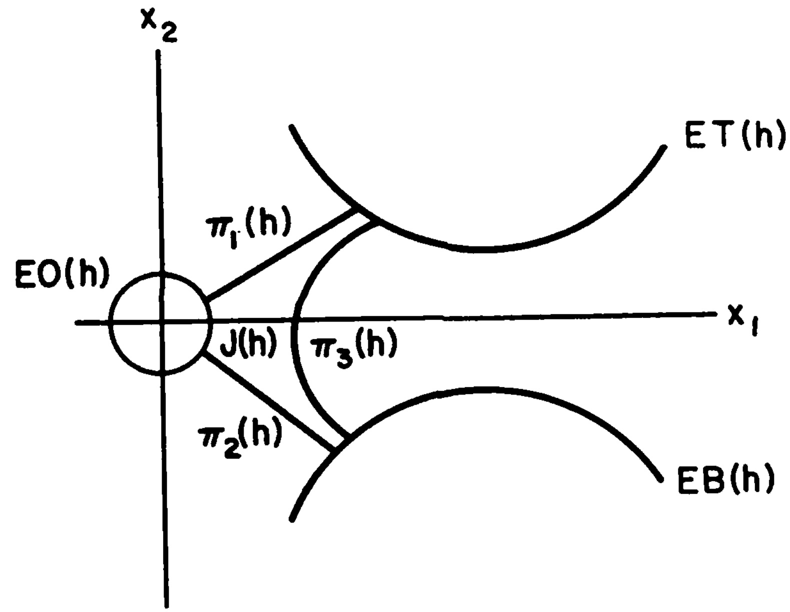


Figure 10

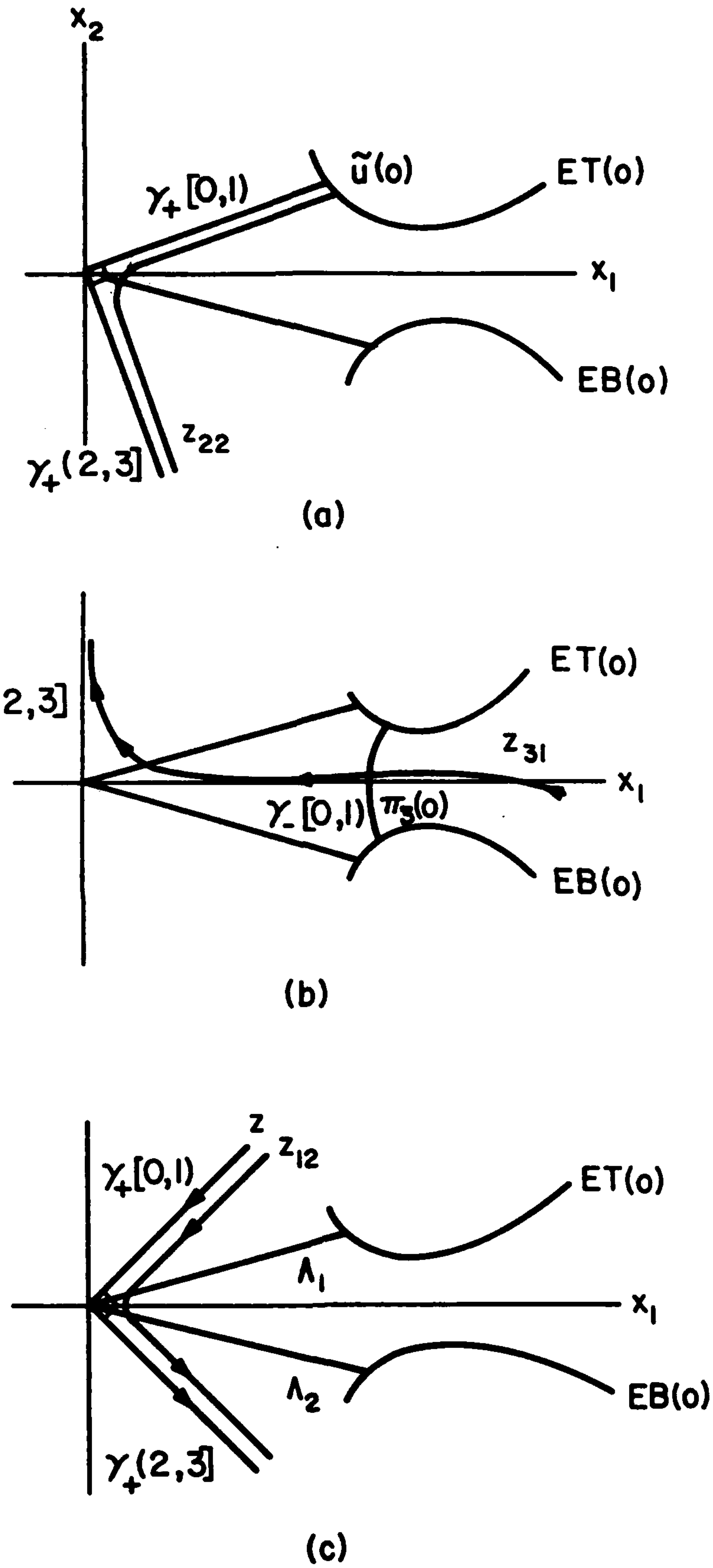

x2
γ+[0,1)
ũ(o)
ET(o)
x1
EB(o)
z22
γ+(2,3]
(a)
γ-(2,3]
ET(o)
z31
x1
γ-[0,1)
π3(o)
EB(o)
(b)
z
z12
γ+[0,1)
ET(o)
Λ1
x1
Λ2
EB(o)
γ+(2,3]
(c)

Figure 11

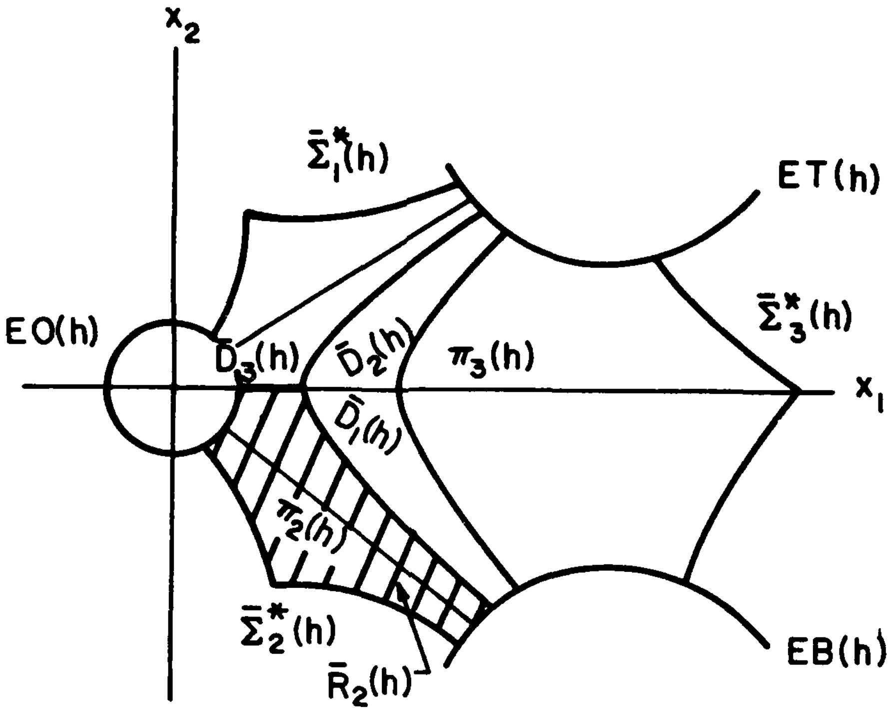


Figure 12

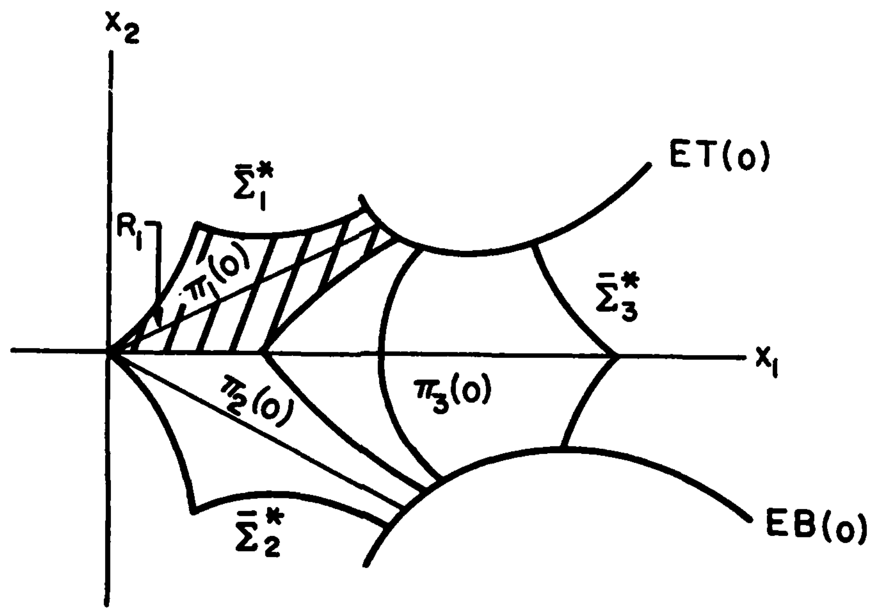


Figure 13

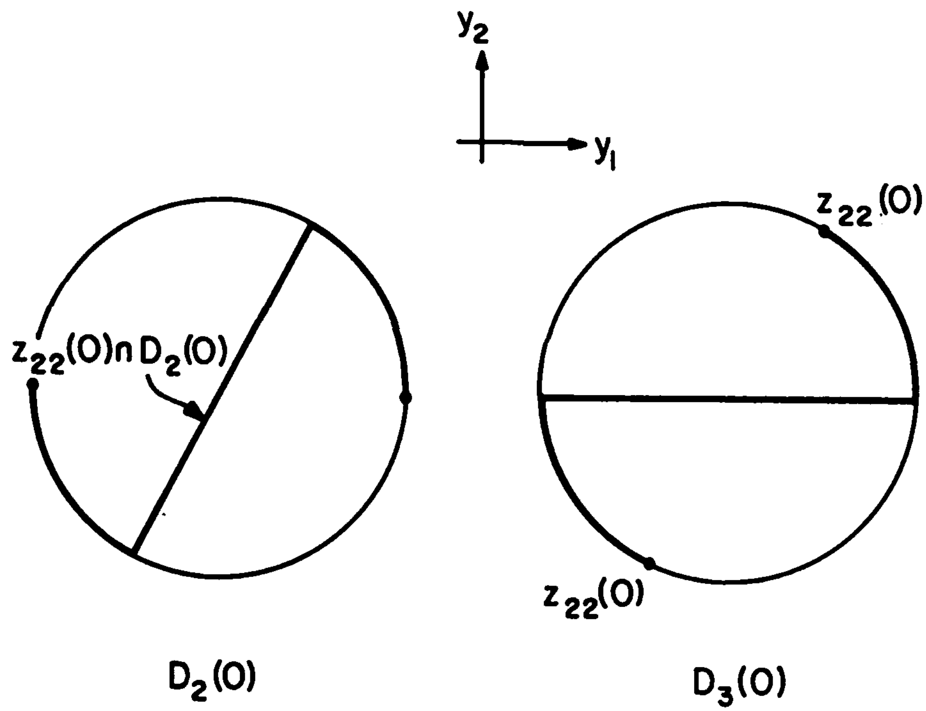


Figure 14

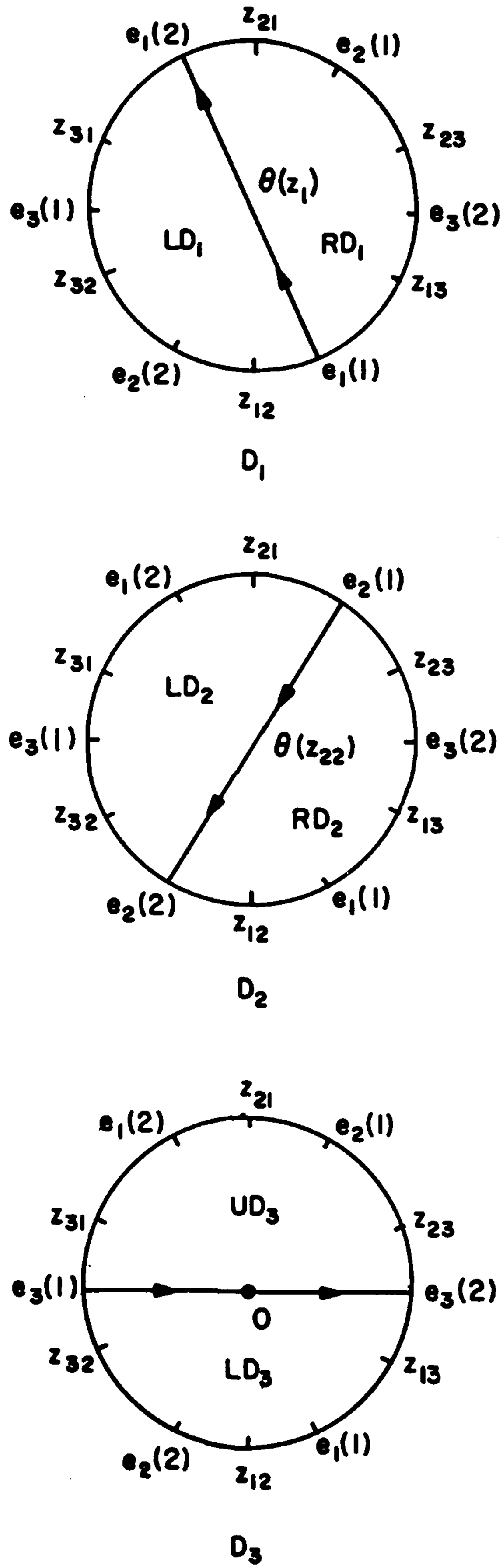
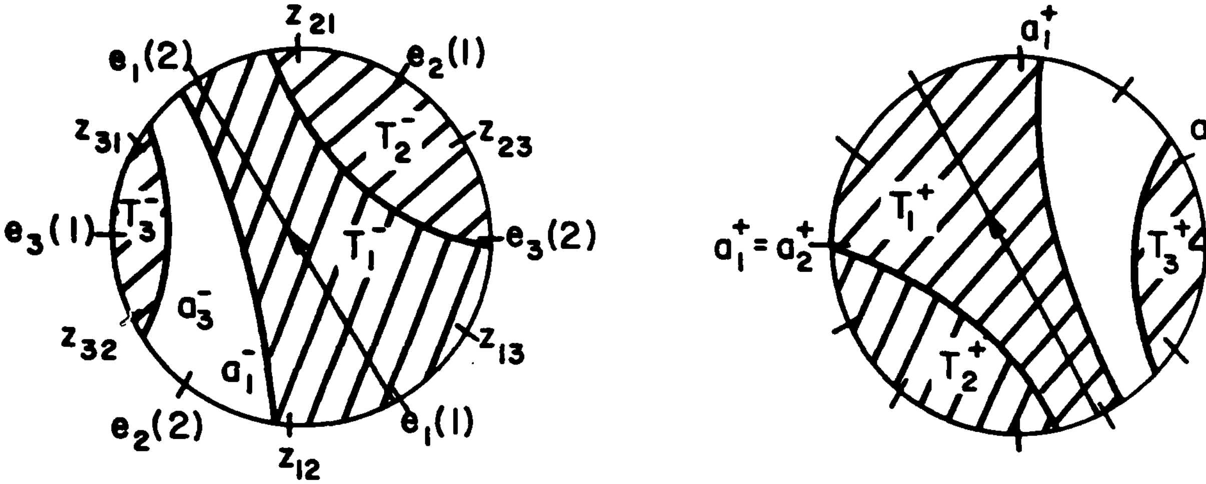
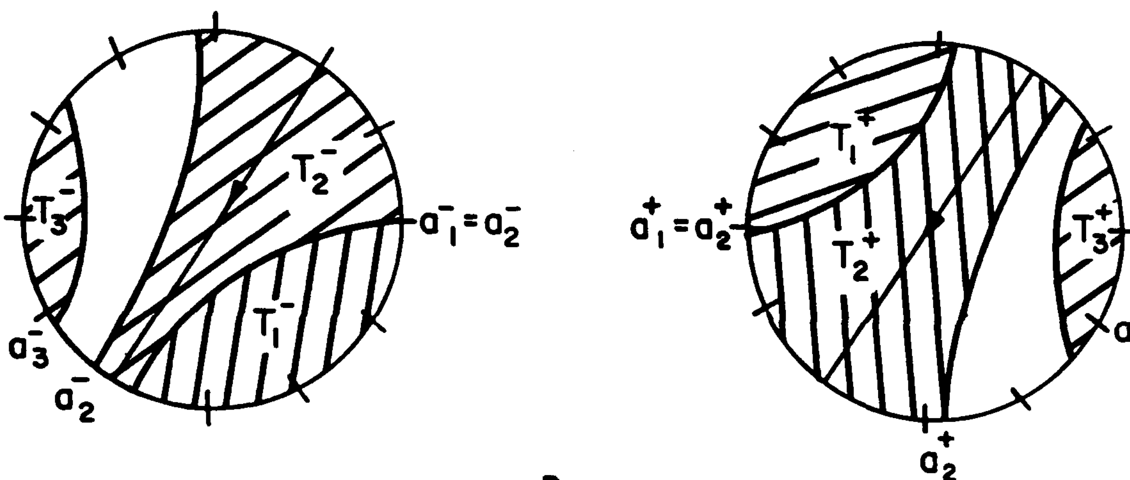

z_{21}
$e_1(2)$
$e_2(1)$
z_{31}
z_{23}
$\theta(z_1)$
$e_3(1)$
$e_3(2)$
LD_1
RD_1
z_{32}
z_{13}
$e_2(2)$
$e_1(1)$
z_{12}
D_1
z_{21}
$e_1(2)$
$e_2(1)$
z_{31}
z_{23}
LD_2
$e_3(1)$
$\theta(z_{22})$
$e_3(2)$
z_{32}
RD_2
z_{13}
$e_2(2)$
$e_1(1)$
z_{12}
D_2
z_{21}
$e_1(2)$
$e_2(1)$
UD_3
z_{31}
z_{23}
$e_3(1)$
$e_3(2)$
O
z_{32}
LD_3
z_{13}
$e_2(2)$
$e_1(1)$
z_{12}
D_3

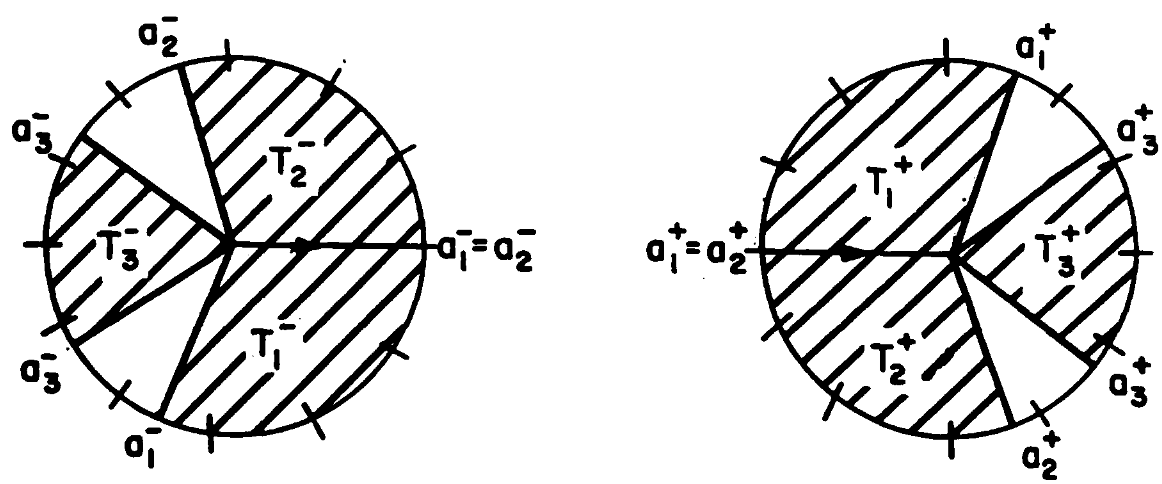
Figure 15



D_1



D_2



D_3

Figure 16

END

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DTIC